[Ap]proved by the Central Text-Book Committee and by the [Gov]ernment of Bengal as a Text, Prize and Library Book.

ELEMENTARY MODERN GEOMETRY

PART I.

(MATRICULATION GEOMETRY)

BOOKS I—IV. *(Complete.)*

[M]eeting the requirements of the Matriculation Examination prescribed under the new regulations of the Calcutta University.

BY

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PREFACE.

"Elementary Modern Geometry, Part I." (Matriculation Geometry) comprises Books I. to IV., which form the Matriculation Course, Part II. containing Book V. (Solid Geometry) only. Each Book has been treated on the lines of the syllabus prescribed under the new regulations of the Calcutta University, and I have spared no pains to keep a steady eye on simplicity and clearness at every stage of the student's progress. Books I., II., III., with the exception of Sections III, and IV. of Book III., may be generally taken to include the Compulsory Course for the Matriculation Examination, whilst Books I. to IV. may be taken to include the Optional (or Additional) Course.

The order in which Propositions are numbered is not the same in every treatise on Modern Geometry. Hence, in all answers meant for the Examiner, the student, while giving a refernce, should *never* quote the number of the Proposition referred to, but simply point ou tits application with clearness and precision. For instance, in proving Theorem 14, Book III, Theorem 10 of the same Book has to be referred to, but all that the student need do is to write out the steps, as given in this treatise, taking care to *omit* "(Th. 10)" at the end of "∴ the $\angle$ ADC = half the $\angle$ AOC."

Model Papers have been inserted at the end of the treatise. The student is strongly recommended to attempt to answer each Paper within the time allotted, and measure his progress by the amount of success attained in such endeavours. It is also advisable that each of these Papers should be answered oftener than once, for a gradually increasing speed and facility may be acquired only by frequent trials.

Any suggestions for the improvement of the work will be thankfully received.

DACCA : *January, 1911.* K. P. BASU.

PREFACE TO THE FIFTH EDITION.

In this Edition, on account of better arrangement and printing the book has increased in volume but in accordance with the wishes of the Hon'ble the Director of Public Instruction, Bengal, the price has been lowered from Re. 1-8 to Re. 1·4 for Books I.—IV. (*Complete*). Books I. & II. and III. & IV. being separately priced at As. 10 and As. 12 respectively. There has also been some improvement in the contents by inserting short references to the various propositions. It is, therefore, hoped that this Edition will be found more useful and acceptable than the previous ones. It is also to be noted that this book has been approved by the Government of Bengal as a Text, Prize and Library Book.

11, Mohendra Gossain Lane, Calcutta,

January, 1916.

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ELEMENTARY

MODERN GEOMETRY

BOOK I.

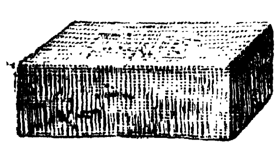
ANGLES AT A POINT; PARALLEL STRAIGHT LINES; TRIANGLES AND RECTILINEAL FIGURES.

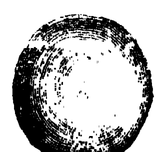
SECTION I.

FUNDAMENTAL IDEAS AND DEFINITIONS.

1. That which has length, breadth and thickness is called a **solid**.

Note 1. Each of the following pictures represents a *solid*:—





Note 2. The first figure represents a brick-shaped body, whilst the second represents a ball. The length, breadth and thickness of the former are easily understood, whilst those of the latter are not. But a ball may be easily conceived to be mostly made up of parts, large and small, which are all brick-shaped. Thus it is clear that anything which may be conceived to be mostly composed of brick-shaped parts is a solid.

Note 3. From the above it is also clear that the hollow portion of a box, quite as much as the entire box, is a solid. In fact, any

definite portion of *space* possessing, as it does, length, breadth and thickness, is a solid.

2. That which has length and breadth, but *no thicknesss*, is called a **surface**.

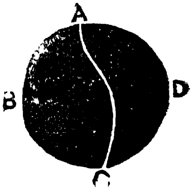
Note 1 A solid is bounded by a *surface* or *surfaces*. For instance, consider a brick-shaped piece of wood. It may be said to have six boundaries, every one of which separates the wood from the rest of space. Each boundary evidently possesses length and breadth, but it has *no thickness*; for, if it had any, it would undoubtedly include a slice of the wood or a thin layer of the surrounding air, or both, which it *does not*. Similarly, the boundary of a ball-shaped body has *no thickness*, because the boundary neither forms part of the body nor that of the surrounding air; but it *has* length and breadth, in the sense that it may be supposed to be divided into small portions, each of which is similar in shape to one or other of the boundaries of a brick-shaped body. The boundary between two portions of the same solid is also a *surface*.

Note 2. It is evident therefore that a surface occupies *no space*.

Note 3. If an ink-spot were made on paper like this would it form a solid or a surface? Undoubtedly it would form a *solid*, because it has not only length and breadth but also *thickness*, however small the thickness may be. Two surfaces of the ink-spot are very close to each other, one (the top-surface) separates the spot from the surrounding air, and the other (the bottom-surface) separates it from the paper.

3. That which has length only, but *neither breadth nor thickness*, is called a **line**

Note 1. The boundary between two portions of a surface is a *line*. Let **ABCD**, in the accompanying diagram, represent a surface and let the white mark **AC** represent the boundary between the portions **ABC** and **ADC**.



So long as the white mark has any breadth at all, the two portions **ABC** and **ADC** form two *distinct* surfaces, one on either side of it; it is only when its breadth dwindles down to *nothing* that the surfaces **ABC** and **ADC** become *parts* of the given surface (*i.e.*, they together make up the whole surface). Thus the boundary between two parts of a surface has length only and *no breadth*, and hence it is a *line*.

Note 2. It is also clear that any given surface is bounded by a line or lines.

Note 3. If an ink-mark like this———be made on paper, it is *not* a line according to our definition, for it has some breadth and also a little thickness. The finer the mark the more nearly does it *resemble* a line.

4. That which has position only, but *neither length nor breadth, nor thickness*, is called a **point**.

Note 1. The boundary between two parts of a given line is a point; because, if this boundary had any length, the two parts would be two distinct lines instead of being parts of the given line. It is clear also that the extremities of a line are points.

Note 2. The picture of a point is a dot (.); but it must be remembered that a dot is in reality a *solid* and *not* a point. The smaller the dot the more nearly does it *resemble* a point.

Note 3. A portion of a line, however small, can *not* be regarded as a point; for, such a portion has some length, whilst a point has none. All that we can say is that the smaller the portion the more nearly does it approach the condition of a point.

Note 4. If a point moves from one position to another very close to it, a small line is generated, of which the two positions of the point are the extremities.

5. Of the three things—length, breadth and thickness, each is called a **dimension**. That which possesses one or more of the three dimensions is called a **magnitude**.

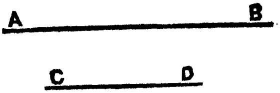
Note 1. Hence a solid is said to be a magnitude of *three* dimensions ; a surface is said to be a magnitude of *two* dimensions ; a line is said to be a magnitude of *one* dimension ; a point is not a magnitude, and it has *no* dimension.

Note 2. The word magnitude is used also in the sense of *size* or *extent*, as in the expression "A point is that which has no parts, or which has no magnitude."

6. If two lines be such that whenever they coincide in two points they invariably become indistinguishable from one another, each of them is called a **straight line** or a **right line**.

Let AB, CD be two lines, as in the following diagram.

Suppose the line CD to be taken up and placed in such a way that the points C and D fall upon the line AB; on whatever points of the line AB the points C and D may fall, if in every case the two lines become indistinguishable one from the other, then *each* of the lines AB and CD is a straight line. In the following diagram the two lines AB and CD are quite distinguishable from one another even though they coincide in two points :—



In this case therefore the lines AB and CD are *not* straight lines.

Note 1. From the above definition the following conclusions are evident :—

(1) Two straight lines cannot have more than one point common ; for, if they had two points common, they would completely become one and the same straight line. This is otherwise expressed

by saying that *two straight lines cannot intersect each other in more points than one.*

(2) Two straight lines cannot coincide partly without being altogether indistinguishable from one another; for, if they coincide partly, they certainly coincide in two points, and therefore they must be indistinguishable throughout. This fact is expressed by saying that *two straight lines cannot have a common segment* (a *segment* of a line meaning a *part* of it).

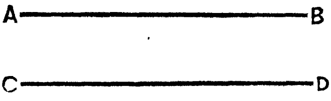
(3) Through one given point an unlimited number of straight lines may pass but *only one straight line can pass through two given points.*

Note 2. A straight line whose extremities are given is called a **finite** or **limited** straight line; whilst one which may be supposed to be extended to any length either way is called an **unlimited** straight line.

Note 3. Let **AB** be a finite straight line whose extremities are the points **A** and **B**.

Let a point **C** start from **A** and move along the line towards **B**. In any position, **C** will divide the line into two parts **AC** and **CB**, and it is quite clear that as **C** moves on, the part **AC** gradually increases whilst **CB** gradually diminishes. Hence there must be one position of **C**, and *one only*, for which the parts **AC** and **CB** will be equal. In this position **C** is said to **bisect** the line **AB**, and it is called the middle point, or, more shortly, the **mid-point** of **AB**. Hence the mid-point of a given straight line may be defined to be the point which divides it into two equal parts.

Note 4. Let **AB** and **CD** be two finite straight lines.

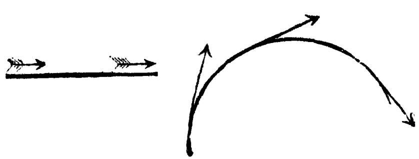


Suppose **AB** is taken up and placed in such a way that the point **A** falls on the point **C** and the line **AB** upon **CD**; if now **B**

also falls on **D**, then the line **AB** *coincides* with **CD** and is therefore equal to it. Hence we may say that *one finite straight line is equal to another when either of them may be made to coincide with the other.*

Note 5. A line which is neither straight nor made up of straight lines is called a **curved line**, or, more simply, a **curve.** The accompanying diagram is the picture of a curved line.

Note 6. When a point moves along a straight line its direction of motion remains unchanged, whilst if a point is moving along a curve its direction of motion constantly changes. In the accompanying pictures the arrow heads shew the directions of motion.



7. If a surface be such that the straight line passing through *any* two points in it lies wholly in that surface, it is called a **plane surface**, or, more simply, a **plane.**

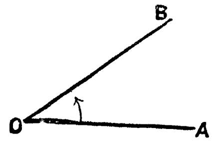
Note 1. If the surface be unlimited all round, the straight line passing through the two points should also be considered as unlimited in both directions. This unlimited straight line then, for all position of the points, will be *completely* in contact with the surface, *if it is a plane.*

Note 2. If two points be taken on the surface of a ball-shaped body, the straight line passing through them does not at all lie in that surface. Clearly therefore such a surface *is not* a plane surface.

Note 3. Henceforth, unless the contrary is stated, *all points and lines shall be supposed to be existing in one and the same plane.*

8. If two straight lines **OA** and **OB** meet at a point **O**, the straight line **OB** may be supposed to have originally

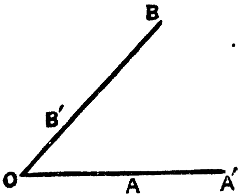
coincided in direction with OA, and then to have moved from that position into its present one by a motion of rotation about the point O; it is this *amount of rotation* which is called the **angle** contained by the straight lines OA and OB.



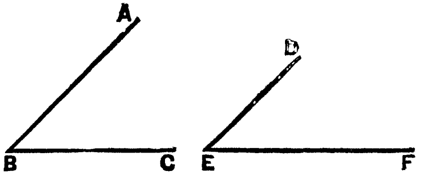
Note 1. *The angle contained by the straight lines* OA *and* OB is briefly expressed as "the angle AOB", or "the angle BOA", the letter O being always put in the middle.

Note 2. The point O is called the vertex of the angle; and the straight lines OA and OB are called its **arms**.

Note 3. The size of an angle does not at all depend upon the *lengths* of its arms. Hence, in the accompanying diagram, the angle AOB is the same as the angle A′OB′.

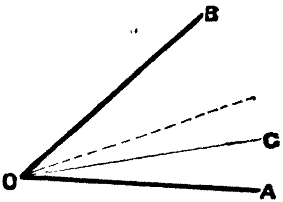


Note 4. Let there be two angles ABC and DEF.



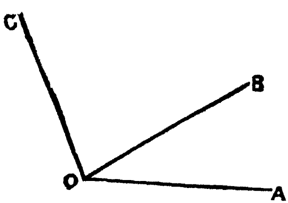
Suppose the angle ABC to be taken up, with its size unchanged, and so placed that the point B falls on the point E, and BC upon EF; BA and ED being on the same side of EF. If now BA also falls upon ED, then the angle ABC *coincides* with the angle DEF and is therefore *equal* to it. Hence *two angles are equal if either of them may be made to coincide with the other.* It is also clear that BA will fall on the left or on the right of ED according as the angle ABC is greater or less than the angle DEF.

Note 5. Let **AOB** be an angle; and let a straight line **OC** starting from a position of coincidence with **OA** rotate about **O** towards **OB**.

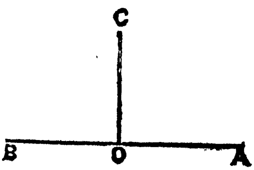


As **OC** goes on rotating, the angle **AOC** gradually increases whilst the angle **COB** gradually decreases. Clearly therefore there must be one position of **OC**, and *one only*, in which the angle **AOC** is *equal* to the angle **COB**. In this position **OC** is said to **bisect** the angle **AOB**, and is called the **bisector** of the angle. Hence the angle **AOB**, and similarly every other angle, *has one and only one bisector*.

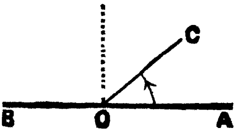
Note 6. If three straight lines **OA**, **OB**, **OC** meet at a point **O**, **OB** occupying an intermediate position, then the angles **AOB** and **BOC** are called **adjacent angles**. In other words, two angles are said to be *adjacent* when they have a common vertex and lie on opposite sides of a common arm. The two extreme lines **OA** and **OC** may be conveniently called the **non-coincident arms** of the two angles.



9. If a straight line **OC** standing upon another straight line **AOB** make the adjacent angles **AOC**, **COB** equal to one another, each of these angles is called a **right angle**; and each of the two straight lines is said to be **perpendicular** to the other.



Note 1. Let **AOB** be any straight line, and **OC** another which starting from the position **OA** rotates about **O** in the direction of the arrow-head, as shewn in the accompanying diagram.

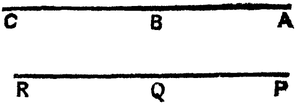


As **OC** goes on rotating, it is evident that the angle **AOC** gradually increases whilst the angle **COB** gradually decreases. Clearly therefore there is one position of **OC**, *and one only*, for which the angle **AOC** is equal to the angle **COB**; and it is in this position that **OC** is perpendicular to **AB**. Hence at any given point in a straight line there can be *only one perpendicular* to it.

Note 2. As **OC** rotates about the point **O** from the position **OA** into the position **OB**, it turns through two right angles. Hence, when the straight lines **OA** and **OB** are in the same straight line but *in opposite directions*, the angle **AOB** is equal to *two right angles*.

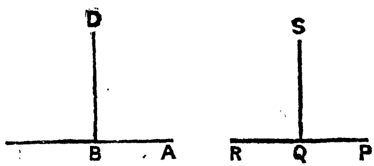
Note 3. When **OA** and **OB** are in the same straight line but in opposite directions, the angle **AOB** is called a **straight angle.** In other words, an angle is called a straight angle *when its arms are in the same straight line but in opposite directions*, and *a straight angle is equal to two right angles.* Hence a *right angle is half of a straight angle.*

Note 4. Let **ABC** and **PQR** be any two straight lines.



Suppose the straight line **ABC** to be taken up and so placed that the point **B** falls on the point **Q**, and **BA** upon **QP**. Then **BC** must also fall upon **QR**; because otherwise the two straight lines would have a common segment. Thus it is clear that if **ABC** and **PQR** be two *straight angles*, they can be made to coincide. Hence the straight angle **PQR** is equal to the straight angle **ABC.** Similarly every other straight angle is equal to the straight angle **ABC.** Clearly therefore *all straight angles are equal to one another.* Hence as a right angle is half of a straight angle, *all right angles are equal.*

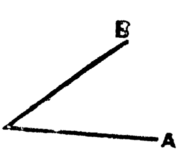
Note 5. That all right angles are equal may also be proved in the following manner. Let the straight line **BD** be perpendicular to the straight line **ABC**, and let the straight line **QS** be perpendicular to the straight line **PQR**. Then we have two pair of right angles **ABD**, **CBD** and **PQS**, **RQS**.



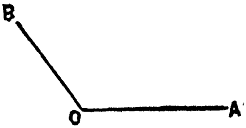
Suppose the first figure to be taken up, without any alteration in its shape, and so placed upon the second that the point **B** falls upon the point **Q**, and the line **ABC** upon the line **PQR**; and so that **BD** and **QS** are on the same side of **PQR**. In this circumstance **BD** and **QS** both become perpendicular to **PQR** at **Q**; therefore **BD** must fall *upon* **QS**, as there can be but *one* perpendicular to a straight line at any point in it. Hence the angle **ABD** is equal to the angle **PQS**, and the angle **CBD** to the angle **RQS**. Hence it is clear that each angle of the first pair is equal to each angle of the second. Similarly, to whichever pair of right angles a particular right angle may belong, this latter must be equal to one or other of the angles **PQS** and **RQS**. Hence, *all right angles are equal to one another.*

Note 6. When we say that **OC** is *at right angles* to **OA** we mean that **OC** is *perpendicular* to **OA**.

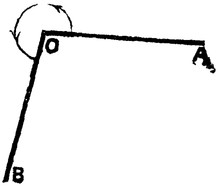
10. An angle which is less than a right angle is called an **acute angle.**



11. An angle which is greater than one right angle but less than two right angles, is called an **obtuse angle.**



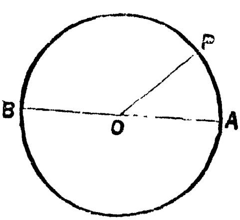
Note. An angle which is greater than two right angles but less than four right angles, is called a **reflex** or **re-entrant angle.**



12. Any portion of a plane surface bounded by one or more lines is called a **plane figure.**

Note. In a more extended sense, any combination of points and lines is called a *figure.* The picture of a figure is called a *diagram.* Real points and lines can never be constructed; so whatever is drawn on paper is only a *picture* of the real thing. From this it is clear that when a figure is drawn on paper it is the diagram that is really drawn, and the finer the diagram the more nearly does it represent the figure intended.

13. A **circle** is a plane figure bounded by one line which is such that all straight lines drawn to it from a certain point within the figure are equal to one another.



14. The bounding line of a circle is called its **circumference.**

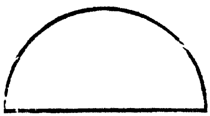
15. The point within a circle from which all straight lines drawn to the circumference are equal, is called the **centre** of the circle.

16. Any straight line drawn from the centre of a circle to the circumference is called a **radius** of the circle.

Note. All radii of the same circle are equal to one another.

17. Any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a **diameter** of the circle.

18. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference which it cuts off.



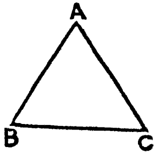
Note. Every diameter divides a circle into two semi-circles.

19. A plane figure which is bounded by straight lines only is called a **plane rectilineal figure.**

Note 1. The bounding straight lines are called the **sides** of the figures.

Note 2. The amount of surface enclosed by the boundaries of a plane figure is called its **area**.

20. A plane figure bounded by *three* straight lines is called a **triangle.**

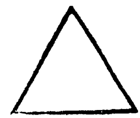


Note 1. Generally speaking, any angular point of a triangle is called a **vertex** of the triangle, so that a triangle has *three vertices.* But when a particular side is spoken of as the **base of** the triangle,

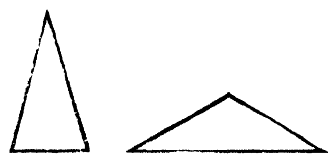
it is then the opposite angular point alone that is called the vertex. Thus, in the above diagram, when the side BC is regarded as the base of the triangle, the angular point A is called the vertex.

Note 2. In any triangle the straight line drawn from a vertex to the middle point of the opposite side is called a **median**.

21. A triangle whose three sides are equal is called an **equilateral triangle.**

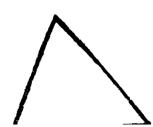


22. A triangle which has two sides equal is called an **isosceles triangle.**

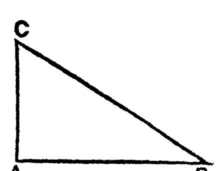


Note. In an isosceles triangle the point of intersection of the equal sides is generally taken as the vertex of the triangle, and the opposite side as the base. The angle contained by the equal sides is called the *vertical angle* of the triangle.

23. A triangle which has three unequal sides is called a **scalene triangle.**

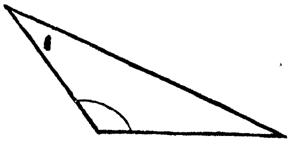


24. A triangle one of whose angles is a right angle is called a **right-angled triangle.**

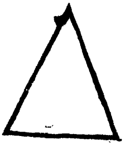


Note. In a right-angled triangle, the side opposite to the right angle is called the **hypotenuse.** Thus, in the above diagram, the side **BC** is the **hypotenuse** of the triangle **ABC** which is right-angled at **A.**

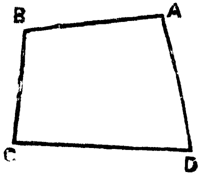
25. A triangle one of whose angles is an obtuse angle is called an **obtuse-angled triangle.**



26. A triangle of which all the angles are acute is called an **acute-angled triangle.**



27. A plane figure bounded by *four* straight iines is called a **quadrilateral.**



Note. A striaght line joining two opposite angular points of a quadrilateral is called a **diagonal** of the quadrilateral. Thus, in the above diagram the straight line joining **A** and **C** is a diagonal, and so is the straight line joining **B** and **D.**

28. A plane figure bounded by *more* than *four* straight lines is called a **polygon.**

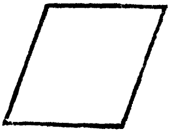
29. Straight lines which are in the same plane and which do not meet, however far they may be produced either way, are called **parallel straight lines.**

30. A quadrilateral whose opposite sides are parallel is called a **parallelogram.**

31. A parallelogram of which one angle is a right angle is called a **rectangle.**

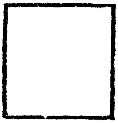
Note. It will be proved hereafter that if one angle of a parallelogram be a right angle *all* its angles are right angles.

32. A parallelogram of which two adjacent sides are equal, but whose angles are not right angles, is called a **rhombus.**

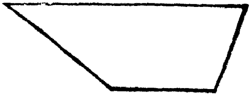


Note. It will be proved hereafter that if two adjacent sides of a parallelogram be equal, *all* its sides are equal.

33. A rectangle of which two adjacent sides are equal, is called a **square.**



34. A quadrilateral which has one pair of opposite sides parallel, is called a **trapezium.**



SECTION II.

POSTULATES AND HYPOTHETICAL CONSTRUCTIONS.

1. For the purpose of effecting geometrical constructions, certain constructions that may be easily admitted as possible are made the foundation of others which are not so obvious. These simple and easily admissible constructions are called **postulates.** They are three in number :—

(1) A straight line may be drawn from any one point to any other point.

(2) A finite (or terminated) straight line may be produced to any length in that straight line.

(3) A circle may be drawn with any point as centre and with a radius equal to any finite straight line.

Note. The first two constructions may be practically carried out with the help of an ungraduated flat ruler, whilst the third requires the use of a pair of compasses which may be so adjusted as to transfer a distance from one position to another.

2. Some constructions which are not so obvious as the postulates, and which, in fact, may have to be performed with the help of one or more of them, may as well be *taken for granted* for the purpose of establishing a geometrical truth. A construction, which is thus assumed before it is shewn *how* it can be performed, is called a **hypothetical construction.** It is clear therefore that a construction which is hypothetical

in one place is not necessarily so in another. In Book I. the following constructions will be assumed before they are formally proved :—

(1) A perpendicular may be drawn to a given straight line from any given point in that straight line.

(2) A straight line may be drawn to bisect any given angle.

(3) Through any given point a straight line may be drawn parallel to a given straight line.

SECTION III.

AXIOMS (OR SELF-EVIDENT TRUTHS).

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are doubles of the same thing, or of equal things, are equal to one another.

7. Things which are halves of the same thing, or of equal things, are equal to one another.

8. Magnitudes which can be made to coincide with one another are equal.

Note 1. It is assumed that any magnitude may be taken up from its position and placed upon another magnitude *without change of shape or side.*

Note 2. The word magnitude is to be understood to mean a line or an angle, or a figure.

Note 3. The process of taking up one magnitude from its position and placing it upon another is called **superposition**; and the former magnitude is said to be **applied** to the latter.

9. The whole is greater than its part.

N B. It is also obvious that the whole is equal to the sum of its patrs.

10. Two straight lines which coincide in their extremities coincide completely.

N. B. This is equivalent to saying that "two straight lines cannot enclose a sarface".

11. All right angles are equal.

12. Two straight lines which intersect one another, cannot be both parallel to the same straight line.

(*Playfair's Axiom.*)

EXERCISE (1).

1. Is a very small ink-spot on paper a point ? if not, what is it ?

2. Is a straight line drawn on paper really a geometrical *straight line* ? If not, what is it ?

3. If a finite straight line be supposed to be divided into a very large number of equal parts, has any of these parts any *dimension* ? If so, what ?

4. What kind of magnitude is the *boundary* between the calm water of a tank and the air above ? and why ?

5. Define a *plane.* If on a finite plane two parallel straight lines be supposed to be drawn *very close* to each other, and to be terminated by the boundary or boundaries of the plane, does any portion of the plane lie between the two lines ? If so, under what circumstances will there be *no* portion of the surface between them ?

6. If two plane surfaces have two points A and B in common, prove that the straight line drawn from A to B is also common to the two planes.

7. If the extremities of one finite straight line fall upon those of another, two lines are equal. Why?

8. If the paper on which a straight line AB is drawn be so folded that the point A comes upon the point B, and if C be the point on AB through which the *crease* passes, then C is the middle point of AB. Why?

9. When does one angle coincide with another?

10. If the paper on which a straight line AB is drawn be so folded that one portion of the line falls upon another, and if O be the point on AB through which the crease passes, and if C be any other point on the crease, prove that OC is perpendicular to AB.

11. If a straight line starting from the position OA continually rotates about O in the same direction until it comes back to its old position, it turns through four right angles. How?

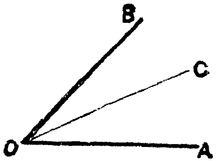
12. AB and CD are two finite straight lines of which AB is greater than CD. Shew how the third postulate can be applied in cutting off from AB a part equal to CD.

13. Shew how the postulates can be applied in drawing from a given point A a straight line equal to a given straight line BC.

14. Two finite straight lines AB and CD are equal to one another. If E be a point on AB and F be a point on CD such that AE is equal to CF, then EB is also equal to FD. Quote the axiom by which this is proved.

15. If one angle ABC be applied to another angle PQR so that B coincides with Q and BC falls upon QR, it is found that BA occupies an intermediate position between QR and QP. Quote the axiom by which it is proved that the angle ABC is less than the angle PQR.

16. AOB is an angle. If the paper be folded so that OA falls on OB and if the straight line OC marks the crease, then OC is the bisector of the angle AOB. How?



SECTION IV.

EXPLANATIONS OF TERMS, SYMBOLS AND ABBREVIATIONS.

1. A complete discussion of a geometrical truth, or of a geometrical construction, is called a **Proposition**.

2. Propositions are of two kinds, *Theorems* and *Problems*

3. A proposition in which a geometrical truth is stated and proved is called a **Theorem.**

4. A proposition in which a geometrical construction is proposed and effected is called a **Problem.**

5. A proposition consists of *four* parts :—The *Enunciation*, the *Particular Enunciation*, the *Construction* and the *Proof*.

(i) The statement in general terms of what is to be proved, or of what is to be done, is called the **Enunciation.**

(ii) The statement of what is to be proved, or of what is to be done, *with special reference to a particular diagram*, is called the **Particular Enunciation.**

(iii) The drawing of such straight lines and circles as may be required to prove the truth of a theorem, or to accomplish the object of a problem, is called the **Construction.**

(iv) The part of a proposition in which, when it is a theorem, the truth of the theorem is proved, or in which

when it is a problem, the construction made is proved to have accomplished the desired end, is called the **Proof.**

Note. In the case of a theorem, the construction may be regarded as forming part of the Proof.

6. The following symbols and abbreviations may be used in writing out propositions :—

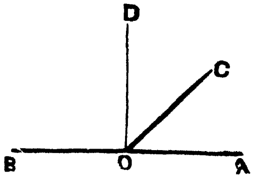
$\therefore$	*for*	therefore.
$\because$	,,	because.
$=$	,,	equal to ; is, or are, equal to.
$\angle$	,,	angle.
rt. $\angle$	,,	right angle.
$\triangle$	,,	triangle.
$\parallel$	,,	parallel.
$\perp$	,,	perpendicular.
$\odot$	,,	circle.
$\odot^{ce}$	,,	circumference
$>$	,,	greater than ; is or are, greater than.
$<$	,,	less than ; is, or are, less than
Pt.	,,	Point.
Str.	,,	Straight line
Par^{ll}.	,,	Parallelogram.
Sq.	,,	Square.
Rect.	,,	Rectangle.
Rectil.	*for*	Rectilineal.
Diff.	,,	Defferent.
Opp.	,,	Opposite.
Adj.	,,	Adjacent.
Diag.	,,	Diagonal.
Def.	,,	Definition.
Post.	,,	Postulate.
Ax.	,,	Axiom.
Cons.	,,	Construction
Hyp.	,,	Hypothesis.
Equilat.	,,	Equilateral.
Isos.	,,	Isosceles.
Quadl.	,,	Quardrilateral.
Reqd.	,,	Required.
Int.	,,	Interior.
Ext.	,,	Exterior.
Prop.	,,	Proposition.
Th.	,,	Theorem.
Cor.	,,	Corollary.
Join AB	,,	Draw the straight line from A to B.

SECTION V.

THEOREMS.

Theorem 1. (EUCLID I. 13.)

If two adjacent angles be such that their non-coincident arms are in the same straight line, then these two angles are together equal to two right angles.



Let AOC and COB be two adjacent angles, of which the non-coincident arms OA and OB are in the same straight line.

To prove that the angles AOC and COB are together equal to two right angles.

Cons. Suppose OD is drawn perpendicular to the straight line AOB.

Proof. Then the $\angle$s AOC, COB are together

= the three $\angle$s AOC, COD, DOB.

Also the $\angle$s AOD, DOB are together

= the same three $\angle$s AOC, COD, DOB.

$\therefore$ the $\angle$s AOC, COB are together

= the $\angle$s AOD, DOB (*Ax. 1.*)

= two right angles. Q. E. D.

Alternative proof.

The $\angle$s AOC, COB are together
= the *straight angle* AOB
= two right angles. (*Def. 9, Note 3.*)

Q. E. D.

Corollary 1. *If two straight lines cut one another, the four angles thus formed are together equal to four right angles.*

Cor. 2. *If any number of straight lines diverge from a given point, the sum of the consecutive angles so formed is equal to four right angles.*

For, if any one of the lines be produced through the common point, the sum of the angles on *each* side of this line = two rt. $\angle$s; and $\therefore$ the sum of the $\angle$s on both sides = four right angles.

Note 1. The letters Q. E. D., which are found at the end of a theorem stand for *Quod erat demonstrandum*, which was to be proved.

Note 2. A geometrical truth which can be easily deduced from a theorem is called a Corollary to it.

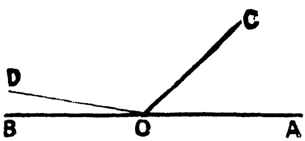
Note 3. In the enunciation of a theorem, that which is *assumed to be true* is called the **Hypothesis**, and that which is *required to be proved* is called the **Conclusion**. In Theorem 1, the *hypothesis* is that *the arms* **OA** and **OB**, of the adjacent angles **AOC** and **COB**, *are in the same straight line;* and the *conclusion* is that *the angles* **AOC** and **COB** *are together equal to two right angles.*

Note 4. When two angles are together equal to two right angles, each of them is called the **Supplement** of the other, and the two angles are said to be **Supplementary** to each other. Thus, in the diagram of Theorem 1, each of the angles **AOC** and **COB** is the *supplement* of the other.

Note 5. When two angles are together equal to one right angle, each of them is called the **complement** of the other, and the two angles are said to be **complementary** to each other. Thus, in the diagram of Theorem 1, each of the angles **AOC** and **COD** is the *complement* of the other.

Theorem 2 (Euc. I. 14.)

If two adjacent angles be such that they are together equal to two right angles, then their non-coincident arms are in the same straight line.



Let the adjacent angles AOC and COB be together equal to two right angles.

To prove that OB **is in the same str. line with** AO.

Proof. OB must be in one of two conditions; it must be either in the same str. line with AO, or not so.

Suppose OB, is *not* in the same str. line with AO, and that OD, which is different from OB, is the produced part of AO.

Now, since OA and OD are in the same str. line, therefore the $\angle$s AOC and COD are together = two right angles.

But, by hypothesis, the $\angle$s AOC and COB are together = two right angles.

$\therefore$ the $\angle$AOC + the $\angle$COD = the $\angle$AOC + the $\angle$COB, (*Ax. 1.*)

$\therefore$ the $\angle$COD = the $\angle$COB; (*Ax. 3.*)

i.e., a part is equal to the whole, which is *impossible*. (*Ax. 9.*)

Thus, it is *absurd* to suppose that OB is *not* in the same straight line with AO.

Clearly therefore OB is in the same str. line with AO.

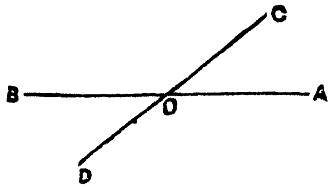
Q. E. D.

Note 1. In Theorem 2, the ***hypothesis*** is that ***the angles* AOC** and **COB** ***are together equal to two right angles***; and the ***conclusion*** is that the ***arms*, OA** and **OB**, ***are in the same straight line.*** Clearly therefore the hypothesis and conclusion of Theorem 2 are respectively the conclusion and hypothesis of Theorem 1. From such a relation between the two theorems, Theorem 2 is said to be the converse of Theorem 1. In other words, if two theorems be so related that ***the hypothesis of each is the conlcusion of the other***, then either of them is said to be the **converse** of the other.

Note 2. The method of proof used in this proposition is called the **Indirect Method.** It is quite clear that the straight line **OB** must *either le* in the same straight line with **AO**, *or not*; hence when the second supposition is proved to be ***absurd***, the inevitable conclusion is that the first is correct. Thus, the ***indirect method*** consists in proving the truth of a theorem ***by proving the absurdity of the supposition that the theorem is not true.*** This method is also called **reductio ad absurdum**; it is generally employed in proving the converse of an estabilshed theorem.

Theorem 3. (Euc. I. 15.)

If two straight lines cut one another, the vertically opposite angles are equal.



Let the straight lines AB, CD cut one another at the point O.

To prove that

(1) the ∠AOC = the ∠BOD
and (2) the ∠COB = the ∠DOA

Proof. The ∠AOC + the ∠COB = two rt. angles
also, the ∠COB + the ∠BOD = two rt. angles; (*Th. 1.*)

∴ the ∠AOC + the ∠COB
= the ∠COB + the ∠BOD. (*Ax. 1.*)

∴ the ∠AOC = the ∠BOD (*Ax. 3.*)

by a similar mode of reasoning, the ∠COB
= the ∠DOA.

Q. E. D.

Cor. *Supplements of the same angle are equal to one another.*

EXCERCISE (2).

1. If two straight lines CC, OD diverge on opposite sides of the straight line AOB, so that the ∠AOC

= the $\angle$ BOD, prove that OD is in the same straight line with CO.

2. Two straight lines AB, CD cut one another at O. Prove that the bisector of the $\angle$ AOC, when produced through O, also bisects the $\angle$ BOD.

3. ABC and PQR are two acute $\angle$s equal to one another. Prove that their *supplements* are equal. Prove also that their *complements* are equal.

4. If two straight lines cut one another and if one of the four angles so formed be a right angle, prove that each of the other three angles is a right angle.

5. If four straight lines OA, OB, OC, OD diverge from a point O, so that each of the $\angle$s AOB, BOC, COD, DOA is a right angle, prove that AO, OC are in the same straight line, and so are BO, OD.

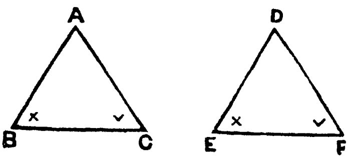
6. Prove that the bisectors of two vertically opposite angles are in one and the same straight line.

7. If four straight lines OA, OB, OC, OD diverge from a point O, so that the $\angle$ AOB = the $\angle$ COD and the $\angle$ BOC = the $\angle$ DOA, prove that AO, OC are in the same straight line, and so are BO, OD.

8. Prove that the internal and external bisectors of an angle are at right angles to each other. (The bisector of an angle and that of its supplement are called *the internal and external bisectors* of the angle.)

Theorem 4. (Euc. I. 4.)

If two triangles be such that two sides and the included angle of one are respectively equal to two sides and the included angle of the other, then these two triangles are equal in all respects.



Let ABC, DEF be two triangles such that

(1) AB = DE,

(2) AC = DF,

and (3) the included $\angle$ BAC = the included $\angle$ EDF.

To prove that the $\triangle$ ABC = the $\triangle$ DEF in all respects.

Proof. Apply the $\triangle$ ABC to the $\triangle$ DEF, so that A falls on D and AB on DE. Then, because the $\angle$ BAC = the $\angle$ EDF. (*Hyp.*)

$\therefore$ AC must fall upon DF.

Now, since AB falls on DE, and AB = DE,

$\therefore$ B falls on E ;

and, since AC falls on DF, and AC = DF,

$\therefore$ C falls on F.

$\therefore$ BC coincides with EF. (*Ax. 10.*)

Thus, the sides AB, AC, BC coincide respectively with DE, DF, EF ;

and $\therefore$ the $\triangle$ ABC coincides with the $\triangle$ DEF.

Hence the $\triangle$ ABC = the $\triangle$DEF *in all respects.*

(*Ax.* 2.)

Q. E. D.

Note 1. Every triangle is said to have six *parts*, the three sides and the three angles. The sides of a triangle also enclose *an amount of surface* which is called its *area*.

Note 2. When one triangle coincides with another, it is evident that all the six parts and the area of the one are respectively equal to all the six parts and the area of the other.

Note 3. In Theorem 4, it is shewn that under the conditions given, the triangles can be made to *coincide* with one another, and hence they are equal *in all respects i.e.*, *all* the six parts and the area of the one are respectively equal to *all* the six parts and the area of the other.

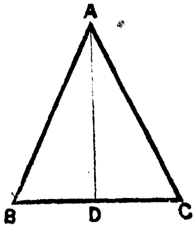
Note 4. When two triangles are equal in all respects, they are said to be **identically equal** or **congruent.**

Note 5. In *congruent* triangles, *angles opposite to equal sides are equal*, as is evident from the above diagram.

Note 6. When we say that the triangles are equal *in all respects*, this certainly includes the three equalities which form the *hypothesis*—AB = DE, AC = DF, $\angle$A = $\angle$D. Hence, to make a clear distinction between hypothesis and conclusion, we should say that the *conclusion* is that BC = EF, $\angle$B = $\angle$E, $\angle$C = $\angle$F, and the area of $\triangle$ ABC = the area of $\triangle$ DEF. That is, of the seven equalities possible, if the above three hold good, the other four must *necessarily* follow.

Theorem 5. (Euc. I. 5.)

If two sides of a triangle are equal, then also the angles opposite to these two sides are equal.



Let ABC be a △, having the side AC = the side AB.

To prove that the ∠ABC = the ∠ACB.

Proof. Let AD be the bisector of the ∠BAC, and let it meet BC in D. Fold the paper about AD from left to right.

Then since the ∠BAD = the ∠CAD,

∴ AB must fall upon AC.

And, since AB = AC, (*Hyp.*)

∴ B must fall on C, and consequently BD will coincide with DC.

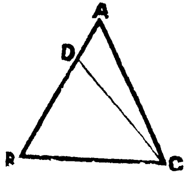
Thus the ∠ABD coincides with the ∠ACD, and is ∴ equal to it. (*Ax. 8.*)

Q. E. D.

Cor. *If a triangle is equilateral, it is also equiangular.*

Theorem 6. (Euc. I. 6.)

If two angles of a triangle are equal, then also the sides opposite to these two angles are equal.



Let ABC be a △, having the ∠ ABC = the ∠ ACB.

To prove that the side AC = the side AB.

Proof. *Suppose* AC is not equal to AB. Then one of them must be the greater; let AB > AC, and from BA cut off BD = AC.

Join DC.

Then in the △s DBC, ACB,

because (1) DB = AC,

(2) BC is common to both,

and (3) the included ∠ DBC

= the inculded ∠ ACB; (*Hyp.*)

∴ the △DBC = the △ACB; (*Th. 4.*)

i.e., a part is equal to the whole,

which is *impossible*. (*Ax. 9.*)

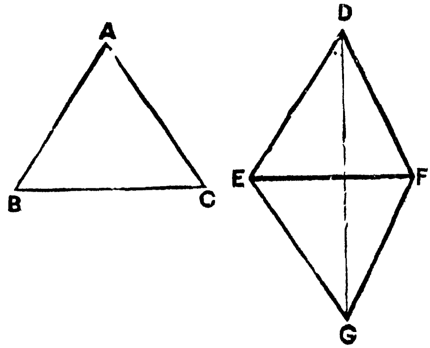
Thus it is *absurd* to suppose that AC is *not* equal to AB. Undoubtedly ∴ AC = AB. Q. E. D.

Cor. *If a triangle is equiangular, it is also equilateral.*

Note. Theorem 6 is evidently the *converse* of Theorem 5.

Theorem 7. (Euc. I. 8.)

If two triangles be such that the three sides of one are respectively equal to the three sides of the other, then these two triangles are equal in all respects.



Let ABC and DEF be two △s, having

$$\left.\begin{aligned} AB &= DE, \\ AC &= DF, \\ \text{and} \quad BC &= EF. \end{aligned}\right\}$$

To prove that the two △s are equal in all respects.

Proof. Of the sides of the △ DEF, let EF be that which is *not less* than either of the other two.

Apply the △ ABC to the △ DEF, so that BC, which = EF, may coincide with EF, and so that A may fall on that side of EF which is remote from D.

Let GEF be the new position of the △ ABC.

Join DG.

Then since $EG = ED$, (*Hyp.*)

$\therefore$ the $\angle EDG =$ the $\angle EGD$; (*Th. 5.*)

and since $FG = FD$, (*Hyp.*)

$\therefore$ the $\angle FDG =$ the $\angle FGD$. (*Th. 5.*)

Hence, the whole $\angle EDF =$ the whole $\angle EGF$;

(*Ax. 2.*)

i.e., the $\angle EDF =$ the $\angle BAC$.

Now in the $\triangle$s ABC, DEF, we have

(1) $AB = DE$,

(2) $AC = DF$,

and (3) the included $\angle BAC =$ the included $\angle EDF$;

$\therefore$ the $\triangle ABC =$ the $\triangle DEF$ in all respects.

(*Th. 4.*)

Q. E. D.

Note. In Theorem 7, it is proved that in the two $\triangle$s **ABC** and **DEF**, if it be *given* that $AB = DE$, $AC = DF$, and $BC = EF$, then it *necessarily* follows that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and the area of the $\triangle ABC =$ the area of the $\triangle DEF$. It must be remembered that ***those angles are equal which are opposite to equal sides.***

EXERCISE (3).

1. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base and is also perpendicular to it.

2. Prove that the straight line drawn from the vertex of an isosceles triangle to the mid-point of the base bisects the vertical angle and is also perpendicular to the base.

3. If BC be the base of an isosceles triangle ABC, and if AB, AC be produced, prove that the angles formed on the other side of the base are equal.

4. AB is a given str. line ; if the str. line CD passes through the mid-point of AB and is also perpendicular to it, prove that *any* point in CD is equidistant from the points A and B.

5. ABC and DBC are two isos. $\triangle$s on the same base BC and on the same side of it. Prove that if AD be joined, and produced to meet the base, it will bisect the vertical angles of the $\triangle$s and be also perpendicular to the base.

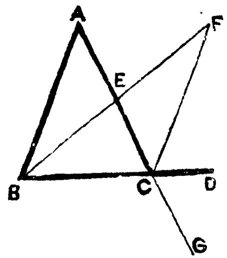
6. ABC is an isos. $\triangle$ on the base BC ; if the bisectors of the base angles meet at D, and if AD be joined, prove that AD bisects the $\angle$ BAC.

7. Prove that the angles opposite to the unequal sides of a triangle are also unequal.

8. If two quadrilaterals ABCD and EFGH be such that AB = EF, BC = FG, CD = GH, $\angle$ ABC = $\angle$ EFG, and $\angle$ BCD = $\angle$ FGH, shew that the two figures can be made to coincide with one another.

Theorem 8. (Euc. I. 16.)

If one side of a triangle be produced, then the exteroir angle is greater than either of the interior opposite angles



Let one side BC of the △ ABC be produced to D.

To prove that the exterior ∠ACD is greater than either of the interior opposite ∠s BAC, ABC.

Let E be the middle point of AC.

Join BE; and produce it to F, making EF = BE.

Join CF.

Proof. In the △s AEB, CEF, we have

(1) AE = CE,

(2) EB = EF,

and (3) the ∠ AEB = the vertically opp. ∠ CEF, (*Th.* 3.)

∴ the two △s are congruent. (*Th.* 4.)

Hence, the ∠ BAE = the ∠ ECF, being opp. to the equal sides BE, EF.

But the ∠ ECD is greater than the ∠ ECF ;

∴ the ∠ ECD is greater than the ∠ BAE ;

i.e., the ∠ ACD is > the ∠ BAC.

In the same way, producing AC to G and joining A to the middle point of BC, it may be proved that the ∠ BCG is > the ∠ ABC.

But the ∠ BCG = the vertically opp. ∠ ACD ;

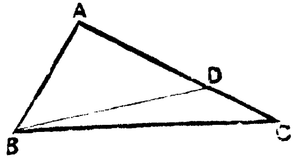
∴ the ∠ ACD is > the ∠ ABC. Q. E. D.

Cor. *Any two angles of a triangle are together less than two right angles.*

For, in the above diagram, the ∠ ABC is less than the ∠ ACD ; ∴ the ∠ ABC + the ∠ ACB is less than the ∠ ACD + the ∠ ACB, and is ∴ < two rt ∠s.

Theorem 9. (Euc. I. 18.)

If one side of a triangle be greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less.



Let ABC be a $\triangle$, having the side AC $>$ the side AB.

To prove that the $\angle$ ABC is $>$ the $\angle$ ACB.

From AC cut off AD = AB.

Join BD.

Proof. Because AB = AD,

$\therefore$ the $\angle$ ADB = the $\angle$ ABD. (*Th.* 5.)

But the ext. $\angle$ ADB of the $\triangle$BDC is greater than the int. opp. $\angle$ DCB, *i.e.*, greater than the $\angle$ ACB.

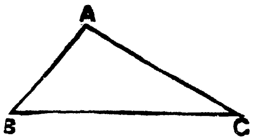
$\therefore$ the $\angle$ ABD is $>$ the $\angle$ ACB.

Much more then is the $\angle$ ABC greater than the $\angle$ ACB.

Q. E. D.

Theorem 10. (Euc. I. 19.)

If one angle of a triangle be greater than another, then the side opposite to the greater angle is greater than the side opposite to the less.



Let ABC be a $\triangle$, having the $\angle$ ABC greater than the $\angle$ ACB.

To prove that the side AC > the side AB.

Proof. If AC is not greater than AB, then it must be *either* = AB, *or* less than AB.

(1) If AC = AB,

then the $\angle$ ABC = the $\angle$ ACB; (*Th.* 5.)

which is impossible, by hypothesis.

(2) If AB be < AB.

then the $\angle$ ABC is less than the $\angle$ ACB, (*Th.* 9.)

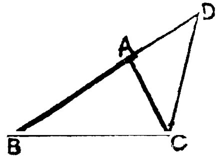
which is also impossible, by hypothesis.

Thus, AC is neither equal to, nor less than AB;

$\therefore$ AC > AB. Q. E. D.

Theorem 11. (Euc. I. 20.)

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

To prove that any two of its sides, AB and AC, are together greater than the third side BC.

Produce BA to D, maikng AD = AC.

Join DC.

Proof. Because AD = AC.

∴ the ∠ ACD = ∠ ADC. (*Th.* 5.)

But the ∠ BCD is > the ∠ ACD ;

∴ the ∠ BCD is also > the ∠ ADC,

that is, the ∠ BCD is > the ∠ BDC.

Hence, from the △BDC, we have
BD greater than BC. (*Th. 10.*)

But BA and AC are together = BD ;

∴ BA and AC are together > BC.

Q. E. D.

Cor. *The difference of any two sides of a triangle is less than the third side.*

For, AB - AC is < BC, if AB < AC + BC, and this latter is true.

Note. A straight line may be regarded as *the shortest distance between its extreme points.* From this point of view, the above theorem is self-evident.

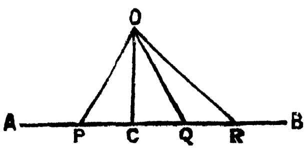
Theorem 12.

Of all the straight lines that can be drawn to a given straight line from a given point outside it,

(i) the perpendicular is the shortest ;

(ii) every two that meet the straight line at equal distances from the foot of the perpendicular, are equal ;

(iii) one that meets the straight line at a greater distance from the foot of the perpendicular than another, is greater than that other.



Let OC be the perpendicular from the given point O to the given str. line AB.

(i) Let OP be *any other* str. line from O meeting the given str. line at P.

To prove that OP > OC.

Proof. Because the side PC of the △ OCP is produced to B,

∴ the ∠ OCB > the ∠ OPC. (*Th. 8.*)

But the ∠ OCB = the ∠ OCP,

∴ the ∠ OCP is > the ∠ OPC.

Hence OP > OC. (*Th. 10.*)

(ii) Let the two str. lines OP and OQ, drawn from the point O to the given str. line, be such that CQ = CP.

To prove that OQ = OP.

Proof. In the $\triangle$s OCP and OCQ, we have

(1) CP = CQ,

(2) CO common,

and (3) the $\angle$OCP = the $\angle$OCQ;

$\therefore$ the two triangles are congruent. (*Th. 4.*)

Hence OQ = OP.

(iii) Let the two str. lines OQ and OR, drawn from the point O to the given str. lines, be such that CR > CQ.

To prove that OR > OQ.

Proof. The ext. $\angle$OQR is > the int. opp. $\angle$OCQ: (*Th. 8.*)

$\therefore$ the $\angle$OQR is *obtuse*.

Also the ext. $\angle$OCA is > the int. opp. $\angle$ORC; (*Th. 8.*)

$\therefore$ the $\angle$ORC is *acute*.

Hence the $\angle$OQR is > the $\angle$ORQ:

$\therefore$ OR > OQ. (*Th. 10.*)

Q. E. D.

Note. The length of the perpendicular from a given point upon a given straight line is called the **distance** of the point from that line.

EXERCISE (4).

1. ABC is a $\triangle$; AD bisects the $\angle$BAC and meets BC at D. Prove that AB > BD, and CA > CD. Hence shew that AB + AC > BC.

2. The two base $\angle$s of any $\triangle$ are together less than two right angles. Prove this by joining the vertex to any point of the base.

3. Prove that in a triangle one angle of which is either right or obtuse, *each* of the other two angles is *acute*.

4. If in a △ ABC, the ∠ BAC be greater than the ∠ ABC, prove that the angle ABC is acute.

5. Prove that only one perpendicular can be drawn to a given straight line from a given point outside it

6. Prove that from a given point outside a given straight line there can be drawn to it two and *only two* straight lines that are equal to one another.

7. D is any point within a △ ABC; if BD and CD are joined, prove that the angle BDC is greater than the angle BAC.

8. D is any point within a △ABC; if BD and CD are joined, prove that AB + AC > BD + DC.

9. If O is any point within a △ ABC, prove that the sum of the lines OA, OB, OC is less than the sum, but greater than half the sum, of the sides of the triangle.

10. If ABCD be a quadrilateral such that AB is the greatest and CD is the least of its sides, prove that the ∠ BCD is greater than the ∠ BAD, and the ∠ ADC is greater than the ∠ ABC.

11. Prove that the sum of the diagonals of any quadrilateral is less than the sum, but greater than half the sum, of the sides of the quadrilateral.

12. AB, AD are two equal sides of a △ABD; if AD is produced to any point C and BC is joined, prove that DC is < BC. Hence deduce a construction for proving that the difference of any two sides of a triangle is less than the third side.

Theorem 13. (Euc. I. 27 and 28.)

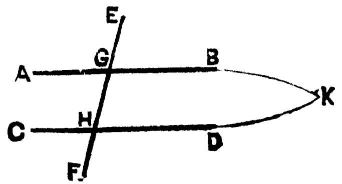
If a straight line, cutting two other straight lines makes,

(i) the alternate angles equal;

or (ii) an exterior angle equal to the interior opposite angle on the same side of it;

or (iii) the interior angles on the same side equal to two right angles;

then those two straight lines are parallel.



Let the str. line EF cut the two str. lines AB, CD at G and H.

(i) Let the alternate $\angle$s AGH, GHD be equal to one another.

To prove that AB and CD are parallel.

Proof. If AB and CD are *not* parallel, they will meet when produced either towards B and D, or towards A and C.

Suppose, when produced towards B and D, they meet at the point K.

Then KGH is a $\triangle$, of which the side KG has been produced to A;

∴ the ext. ∠ AGH is > the int. opp. ∠ GHK. (*Th. 8.*)

But this is contrary to the hypothesis ;

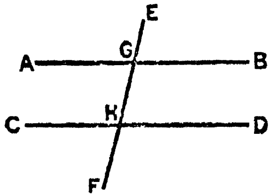
∴ AB and CD cannot meet when produced towards B and D.

Similarly they can neither meet towards A and C.

Hence AB and CD are parallel.

(ii) Let the ext. ∠ EGB be = the int. opp. ∠ GHD.

To prove that AB and CD are parallel.



Proof. Because the ∠ EGB = the ∠ GHD,

and the ∠ EGB = the vertically opp. ∠ AGH ;

∴ the ∠ AGH = the ∠ GHD, (*Ax. 1.*)

which are *alternate* angles.

∴ AB and CD are parallel.

(iii) Let the two int. ∠ s BGH, GHD, on the same side of EF, be together = two rt. ∠ s.

To prove that AB and CD are parallel.

Proof. Because the ∠ BGH + the ∠ GHD
= two rt. ∠ s,

and also the ∠ BGH + the ∠ AGH
= two rt. ∠ s ; (*Th. 1.*)

∴ the ∠ BGH + the ∠ GHD
= the ∠ BGH + the ∠ AGH. (*Ax. 1.*)

Hence, the $\angle$ **GHD** = the $\angle$ **AGH**, (*Ax.* 3.)

which are *alternate* angles.

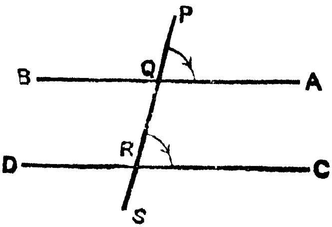
$\therefore$ **AB** and **CD** are parallel. Q. E. D.

Cor. *If two straight lines are perpendicular to the same straight line, they are parallel to one another.*

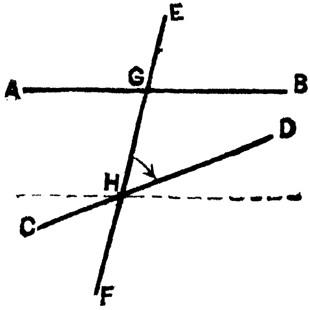
Note 1. Altogether eight $\angle$s have been formed at the points **G** and **H**. Of these $\angle$s, the two that lie above the line **AB**, as well as the two that lie below **CD**, are called *exterior* angles ; whilst the remaining four $\angle$s are called *interior* angles. They are only two pairs of *alternate* $\angle$s ; $\angle$s **AGH** and **GHD** forming one pair, and the $\angle$s **BGH** and **GHC** forming the other. Any ext. $\angle$ and the int. opp. $\angle$ on the same side of **EF** are said to form a pair of *corresponding* $\angle$s ; thus the $\angle$s **EGB** and **GHD** are a pair of *corresponding* $\angle$s ; similarly there are three other pairs of corresponding $\angle$s.

Note 2. There are three different hypothesis, and in each case the conclusion is one and the same. Hence the proposition might very well be split up into three separate propositions.

Note 3. Any straight line that cuts two or more given lines is called a **transversal**. Thus, in the above diagram, the straight line **EF** is a transversal.



Note 4. If any fixed and unlimited str. line **PQRS** cuts any two str. lines **AB**, **CD** at **Q** and **R**, **AB** is ∥ to **CD** when the $\angle$s **AQP** and **CRP** are *equal*. Hence if **AB** and **CD** be both supposed to be initially on the fixed line **PS**, and then to turn in the *watch-hand* way, **AB** about the point **Q** and **CD** about the point **R**, they will become parallel when they have undergone the *same amount of turning*. Hence, two str. lines are parallel when they have the *same direction*.



Note 5. Let a str. line **EF** cut a str. line **AB** at **G**. Let another str. line **CD** rotate about the pt. **H** on **EF**, in the direction of the arrow head, starting from a position of coincidence with **EF**. Then it is evident that the $\angle$**EHD** gradually incereases, whilst the $\angle$**DHF** gradually decreases. Hence there is one position of **CD** and ***one only*** in which the $\angle$**EHD**=the $\angle$**EGB**, and in this position **CD** is parallel to **AB**.

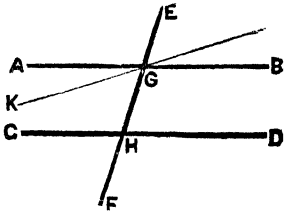
Theorem 14. (Euc. I. 29.)

If a straight line cuts two parallel straight lines, it makes,

(*i*) *the alternate angles equal to one another ;*

(*ii*) *the exterior angle equal to the interior opposite angle on the same side of it ;*

and (*iii*) *the two interior angles on the same side together equal to two right angles.*



Let the str. line EF cut the ∥ str. lines AB, CD at the pts. G and H.

(1) To prove that the ∠AGH = the alt. ∠GHD.

Proof. If the ∠AGH be not equal to the ∠GHD, let the ∠KGH be equal to it.

Now since the ∠s KGH and GHD are alt. ∠s, and equal to one another.

∴ KG is parallel to CD. (*Th. 13.*)

But, by hypothesis, AB is parallel to CD ;

∴ the two str. lines AG, KG, which intersect one another, are *both* ∥ to CD, which is impossible.

(*Playfair's Axiom.*)

Thus, the ∠AGH cannot be unequal to the ∠GHD ;

∴ ∠AGH = the ∠GHD.

(2) To prove that the ext. ∠EGB = the int. opp. ∠GHD.

Proof. Because the $\angle$ AGH

= the $\angle$ GHD, (*already proved.*)

and also the $\angle$ AGH = the vertically opp. $\angle$ EGB ;

$\therefore$ the $\angle$ EGB = the $\angle$ GHD. (*Ax. 1.*)

(3) **To prove that the two interior $\angle$s BGH, GHD are together = two rt. $\angle$s.**

Proof. Because the $\angle$ EGB = the $\angle$ GHD,

(*proved above.*)

$\therefore$ the $\angle$ EGB + the $\angle$ BGH = the $\angle$ GHD + the $\angle$ BGH.

But the $\angle$s EGB, BGH are together

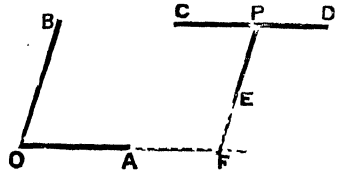
= two rt. $\angle$s ; (*Th. 1.*)

$\therefore$ the $\angle$s BGH, GHD are together

= two rt. $\angle$s. Q. E. D.

Cor. 1. *If a straight line is perpendicular to one of two parallel straight lines, it is also perpendicular to the other.*

Cor. 2. *Two angles having parallel arms are either equal or supplementary.*

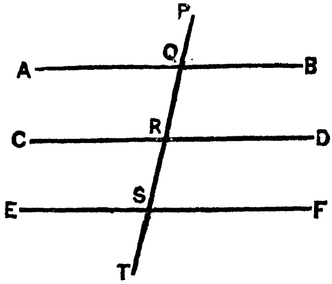


In the accompanying diagram, let the str. line CPD be ∥ to OA and PE to OB. If OA and PE be produced to meet at F, it is easy to see that $\angle$s AOB, CPE are *equal*, each of them being supplementary to the $\angle$ PFO; and that $\angle$s AOB, DPE are *supplementary*, the $\angle$ DPE being = the $\angle$ PFO which is supplementary to the $\angle$ AOB.

Note. In any str. line **AB**, the direction from **A** to **B** is *opposite* to that from **B** to **A**. This idea is also expressed by saying that the line **AB** is opposite in sense to the line **BA**.

Theorem 15. (Euc. I. 30.)

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to EF.

To prove that AB is parallel to CD

Draw any straight line PT cutting AB, CD and EF in the points Q, R and S.

Proof. Since AB is ∥ to EF, and PT cuts them,

∴ the ∠AQS = the alt. ∠QSF. (*Th. 14.*)

Again, since CD is ∥ to EF, and PT cuts them,

∴ the ext. ∠QRD = the int. opp. ∠QSF. (*Th. 14.*)

Hence the ∠AQR = the ∠QRD. (*Ax. 1.*)

But these are alternate ∠s ;

∴ AB is ∥ to CD. Q. E. D.

Alternative Proof. If AB is not ∥ to CD, they will meet when produced ; and then we have two intersecting str. lines both parallel to a third straight line, which is imposible. (*Playfair's Axiom.*)

Hence AB and CD cannot meet, and ∴ they are parallel.

Note. If EF lies *between* AB and CD, the truth of the theorem becomes obvious. For, if AB and CD were to meet, one of them would cut EF, which is impossible. Hence AB and CD cannot meet, and ∴ they are parallel.

EXERCISE (5).

1 If a str. line cuts two parallel str. lines prove that the bisectors of any pair of alt. angles are parallel.

2. If one angle of a parallelogram be a right angle, prove that the other three angles also are right angles.

3. If a straight line intersecting two other straight lines make two exterior angles on the same side of the line together equal to two right angles, prove that the two straight lines are parallel.

4. If through the vertex of an isosceles triangle a straight line be drawn parallel to the base, prove that it bisects the exterior vertical angle.

5. If the side BC of a triangle ABC be produced to D, shew that the bisectors of the angles BAC, ACD cannot be parallel.

6. If two straight lines be such that one of them is perpendicular to a given straight line while the other is not, prove that they must intersect each other.

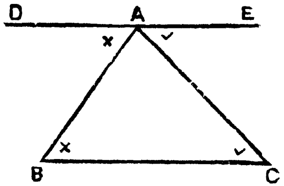
7. If any two straight lines be drawn perpendicular to two given intersecting straight lines, prove that they must interesect.

8. Through each angular point of a triangle a str. line is drawn parallel to the opposite side ; prove that the triangle formed by these three str. lines is equiangular to the given triangle.

9. If a quadrilateral ABCD be such that the diagonals AC, BD bisect each other, prove that the quadrilateral is a parallelogram.

Theorem 16. (Euc. I. 32.)

The sum of the angles of a triangle is equal to two right angles.



Let ABC be a triangle.

To prove that the ∠s ABC, BCA, CAB **are together = two rt.** ∠s.

Suppose, through the pt. A, the str. line DAE has been drawn ∥ to BC.

Proof. Since DE is ∥ to BC and AB meets them

∴ the ∠ ABC = the alt. ∠ DAB.

Again, since DE is ∥ to BC and AC meets them,

∴ the ∠ BCA = the alt. ∠ CAE.

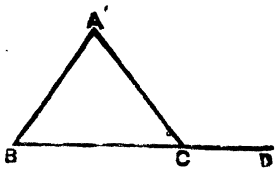
Hence, the ∠s ABC, BCA are together
= the ∠s DAB, CAE.

To each of these equals add the ∠ BAC ;

∴ the sum of the ∠s ABC, BCA, CAB
= the sum of the ∠s DAB, BAC, CAE
= two right angles. Q. E. D.

Cor. 1. *If any side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.*

In the accompanying diagram, the sum of the $\angle$s BCA, ACD = two rt. $\angle$s = the $\angle$BCA + the $\angle$CAB + the $\angle$ABC; $\therefore$ the $\angle$ACD = the $\angle$CAB + the $\angle$ABC. (*Ax. 3.*)



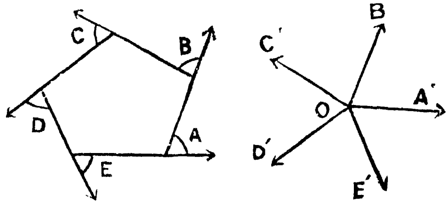
Cor. 2. *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to third angle of the other.*

Note 1. If a right angle be supposed to be divided into 90 equal parts, each part is called a **degree**; a sixtieth part of a degree is called a **minute**, and a sixtieth part of a minute is called a **second**. In other words, 60 seconds make a minute, 60 minutes make a degree, and 90 degrees make a right angle. 8 degrees, 13 minutes and 20 seconds is written as $8^\circ\ 13'\ 20''$.

Note 2. The sum of the angles of a triangle = 180°.

Theorem 17.

If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then the sum of the exterior angles so formed is equal to four right angles.



Let the sides of the rectilineal figure ABCDE be respectively produced in the directions shewn by the arrow-heads.

To prove that the exterior $\angle$s A, B, C, D, E are together = four right $\angle$s.

Take any point O, and suppose the str. lines OA, OB′, OC′, OD′, OE′ are drawn ∥ respectively to the sides EA, AB, BC, CD, DE; each being drawn in the *sense* in which the corresponding side is produced.

Proof. Since OA′ is ∥ to EA produced, and OB′ is ∥ to AB,

∴ the ext. $\angle$ A = the $\angle$ A′OB′. (*Th. 14, Cor. 3.*)

Similarly, the ext. $\angle$ B = the $\angle$ B′OC′,
the ext. $\angle$ C = the $\angle$ C′OD′,
the ext. $\angle$ D = the $\angle$ D′OE′.
the ext. $\angle$ E = the $\angle$ E′O′A′.

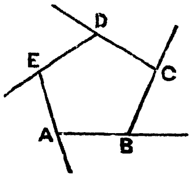
Hence, the sum of the ext. $\angle$s A, B, C, D, E

= the sum of the $\angle$s at O .

= four right angles. (*Th. 1, Cor. 2.*)

Q. E. D.

Cor. ***If a convex* rectilineal figure has n sides, then the sum of all its interior angles is equal to (2n − 4) right angles.***



Let ABCDE be a convex rectilineal figure having n sides ; then it has also n interior $\angle$s and n vertices.

If the sides be produced in order, we find that the sum of the int. and ext. $\angle$s at each vertex = 2 rt. $\angle$s.

Hence, the sum of the int. and ext. $\angle$s at the n vertices $= 2n$ rt. $\angle$s.

That is, the sum of all the int. $\angle$s + the sum of the ext. $\angle$s $= 2n$ rt. $\angle$s.

But the sum of the ext. $\angle$s = 4 rt. $\angle$s ; $\therefore$ the sum of the int. $\angle$s = $(2n - 4)$ rt. $\angle$s.

Note 1. The sides of a rectilineal figure are said to be produced ***in order***, when, considering a point to move round the figure along its sides, each side is produced in the *sense* in which the point is moving along it.

Note 2. A rectilineal figure is said to be **regular** when it is both equilateral and equiangular. A rectilineal figure having five sides is called a **pentagon**, and one having six sides is called a **hexagon**. Hence a regular pentagon, or a regular hexagon, is one which has all its sides and angles equal.

Note 3. The sum of the interior angles of a pentagon = 6 right angles. Hence each angle of a ***regular*** pentagon = one-fifth of 6 rt. angles = 108°. Similarly each angle of a ***regular*** hexagon = 120°.

*A rectilineal figure is said to be *convex* when it has no re-entrant angle.

EXERCISE (6).

1. Prove that in a rt. angled triangle, each of the acute angles is the complement of the other.

2. If one angle of a triangle be equal to the sum of the other two, prove that the triangle is rt. angled.

3. If one angle of a triangle be greater than the sum of the other two, prove that the triangle is obtuse-angled.

4. Prove that the sum of the angles of a quadrilateral is equal to four right angles.

5. If one side of a triangle be produced the exterior angle is equal to the sum of the two interior opposite angles. Shew how this can be proved without deducing it as a corollary to Theorem 16.

6. Prove by the application of Theorem 16, that the interior angles of a pentagon are together equal to six right angles.

7. Find the magnitude of an angle of a regular polygon of ten sides.

8. Prove that the acute angle between two straight lines is equal to the acute angle between any two straight lines at right angles to them.

9 Find the magnitude of the vertical angle of an isosceles triangle each of the base angles of which is double of the vertical angle.

10. Draw a diagram shewing that a heptagon (*i.e.*, a rectilineal figure of seven sides) can be divided into five

triangles, and hence shew that the sum of its interior angles is equal to ten right angles.

11. Prove that the straight line bisecting the exterior vertical angle of an isosceles triangle is parallel to the base.

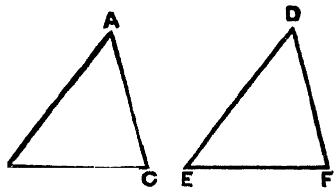
12. ABC, ABD, ACE are equilateral triangles; prove that the points D, A, E are in the same straight tine.

13. A is the vertex of an isosceles triangle ABC, and BA is produced to D, so that AD is equal to AB; if DC is joined, prove that BCD is a right angle.

14. BC is the hypotenuse of a right-angled triangle ABC. If AD be drawn to meet BC, so that the angle BAD is equal to the angle ABD, prove that D is the middle point of BC and that AD is half of BC.

Theorem 18. (Euc. I. 26.)

If two triangles be such that two angles and a side of the one triangle are respectively equal to two angles and the corresponding side of the other, then these two triangles are equal in every respect.



Let the ∠s ABC, BCA of the △ ABC be respectively equal to the ∠s DEF, EFD of the △DEF. Also

(1) let the side BC be = the *corresponding* side EF,

or (2) let the side AB be = the *corresponding* side DE,

or (3) let the side AC be = the *corresponding* side DF.

To prove that in each of these three cases the two △s are congruent.

Case I. Apply the △ ABC to the △ DEF so that B may be on E and BC on EF.

Then, because BC = EF,

∴ C must coincide with F.

Now, since the ∠B = the ∠E,

∴ BA must fall on ED ;

and since the ∠C = the ∠F,

∴ CA must fall on FD.

Thus the point A falls both on ED and FD;

$\therefore$ A falls on D which is the point in which ED and FD intersect.

Hence the two $\triangle$s coincide, and are $\therefore$ equal in every respect.

Case II. Since two $\angle$s of the $\triangle$ABC are respectively equal to two $\angle$s of the $\triangle$ DEF,

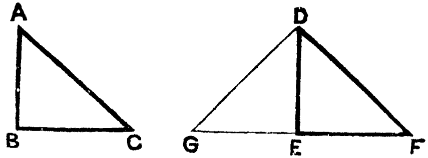
$\therefore$ the remaining $\angle$ BAC of the one
= the remaining $\angle$ EDF of the other.
(*Th. 16, Cor. 2.*)

Thus we have the $\angle$s A, B respectively = the $\angle$s D, E and the side AB = the side DE. From this point we have to proceed as in Case I, by so applying the $\triangle$ABC to the $\triangle$DEF that AB may coincide with DE.

Case III. This is exactly similar to Case II, and we have to proceed by applying the $\triangle$ABC to the $\triangle$DEF so that AC may coincide with DF.

Theorem 19.

Two right-angled triangles which have their hypotenuses equal, and one side of the one equal to one side of the other, are equal in all respects



Let ABC and DEF be two right-angled △s, in which the hypotenuse AC = the hypotenuse DF, and the side AB = the side DE.

To prove that the two △s are equal in all respects.

Proof. Apply the △ABC to the △DEF so that AB may coincide with DE to which it is equal, and C may fall on that side of DE which is remote from F.

Thus, let DEG be the new position of the △ABC.

Now, since each of the ∠s DEF, DEG is a rt. ∠,

∴ EG is in the same str. line with FE. (*Th. 2.*)

Hence DGF is a △, and since DF = DG. (*Hyp.*)

∴ the ∠DGF = the ∠DFG. (*Th. 5.*)

Hence, in the △s DEG, DEF, we have

(1) the ∠DEG = the ∠DEF,

(2) the ∠DGE = the ∠DFE. (*proved above.*)

and (3) DE common;

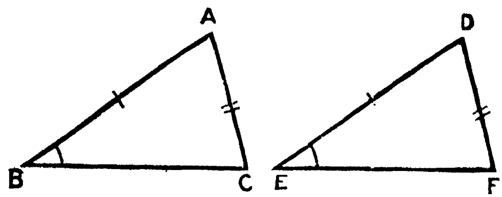
∴ these two △s are equal in all respects. (*Th. 18.*)

That is, the △s ABC, DEF are equal in all respects.

Q. E. D.

Theorem 20.

If two triangles have two sides of the one respectively equal to two sides of the other, and have also the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary.



Let the $\triangle$s ABC, DEF be such that AB = DE, AC = DF, and the $\angle$ABC = the $\angle$DEF.

To prove that the $\angle$s ACB, DFE are either equal or supplementary.

Proof. The angle BAC must be *either* equal to the $\angle$EDF *or* not.

Case I. When the $\angle$BAC = the $\angle$EDF.

In the $\triangle$s ABC, DEF, we have

(i) AB = DE,

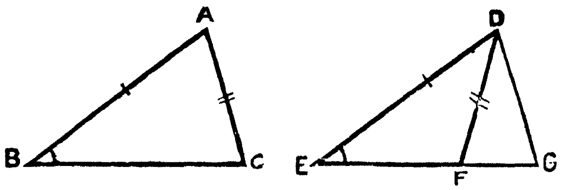
(ii) AC = DF,

and (iii) the $\angle$BAC = the $\angle$EDF;

$\therefore$ the two $\triangle$s are congruent. (*Th. 4.*)

Hence the $\angle$ACB = the $\angle$DFE.

Case II. When the $\angle$ BAC is *not* equal to the $\angle$ EDF.



On the same side of DE as F, suppose the $\angle$ EDG is made equal to the $\angle$ BAC, and let EF, produced if necessary, meet DG in G.

Then in the $\triangle$s ABC, DEG, we have

(1) the $\angle$ ABC = the $\angle$ DEG,

(2) the $\angle$ BAC = the $\angle$ EDG, (*Cons.*)

and (3) AB = DE;

$\therefore$ the two $\triangle$s are congruent. (*Th. 18.*)

Hence, DG = AC = DF;
and also the $\angle$ DGE = the $\angle$ ACB.

Hence the $\angle$ DFG = the $\angle$ DGE = the $\angle$ ACB.

But the $\angle$ DFG is supplementary to the $\angle$ DFE;

$\therefore$ the $\angle$ ACB is supplementary to the $\angle$ DFE.

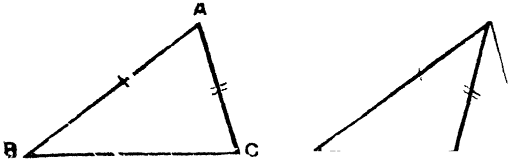
Q. E. D.

Note. It is possible for the angle **ACB** and **DFE** to be supplementary *only when one of them is acute and the other obtuse.*

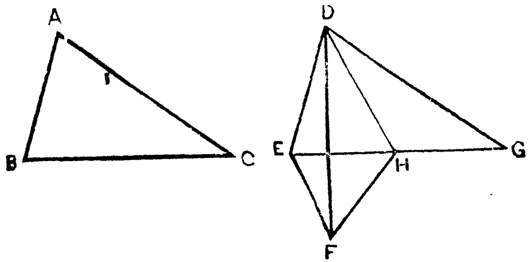
Theorem 21. (Euc. I. 24.)

If two triangles be such that two sides of the one are respectively equal to two sides of the other, but the angle contained by the two sides of the one is greater than the angle contained by the two sides of the other, then the base of that which has the greater angle is greater than the base of the other.

(1)



(2)



Let ABC, DEF be two $\triangle$s such that

$$AB = DE,$$
$$AC = DF,$$

but the $\angle$ BAC is greater than the $\angle$ EDF.

To prove that the base BC is greater than the base EF.

Proof. Apply the $\triangle$ABC to the $\triangle$DEF so that A may fall on D and AB on DE.

Then since A coincides with D and AB = DE ∴ B coincides with E.

Let DG, EG be the new position of AC, BC.

(1) If EG passses through F as in fig. (1), then EG is > EF ; *i.e.*, BC is > EF.

(2) If EG does not pass through F, as in fig. (2), suppose the ∠ FDG is bisected by DH which meets EG at H.

Join FH.

Then in the △s FDH, GDH, we have

(i) FD = GD,

(ii) DH common,

and (iii) the ∠ FDH = the ∠ GDH ; (*Cons.*)

∴ the two triangles are congruent. (*Th. 4.*)

Hence, FH = GH,

and ∴ EG = EH + HF.

But the two sides EH, HF of the △EHF are together > EF ;

∴ EG > EF,

that is, BC > EF. Q. E. D.

Cor. *If two triangles* ABC, DEF *be such that* AB *is equal to* DE, AC *is equal to* DF, *but the base* BC *is greater than the base* EF *then the angle* BAC *is greater than the angle* EDF.

The ∠ BAC cannot be less than the ∠ EDF ; for then BC would be less than EF, which is impossible.

Nor can the ∠ BAC be equal to the ∠ EDF ; for then BC would be equal to EF, which also is impossible.

∴ the ∠ BAC *must be* > the ∠ EDF.

EXERCISE (7).

1. If the bisector of the vertical angle of a triangle is also perpendicular to the base, prove that the triangle is isosceles.

2. Prove that any point on the bisector of an angle is equidistant from the arms of the angle.

3. ABC is a triangle of which the side AB is greater than the side AC ; if D is the mid-point of BC and if AD is joined, prove that the angle ADB is an obtuse angle.

4. If a straight line AB is bisected at O, prove that the perpendiculars from A and B upon any other straight line through O are equal.

5. Prove that perpendiculars drawn from the extremities of the base of an isosceles triangle upon the opposite sides are equal.

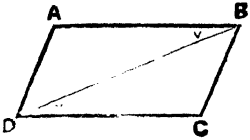
6. OB and OC are perpendiculars upon two straight lines AB and AC which meet at A ; if $OB = OC$, prove that AO is the bisector of the angle BAC.

7. ABC is a triangle of which the side AB is greater than the side AC; if D is the mid-point of BC, and if any point E be taken on AD, prove that BE is greater than CE.

8. ABCD is a quadrilateral such that the diagonal AC bisects each of the angles BAD, BCD. Prove that AC bisects BD at right angles,

Theorem 22. (Euc. I. 33.)

If a quadrilateral be such that two of its opposite sides are equal and parallel, then the other two sides also are equal and parallel.



Let ABCD be a quadrilateral such that two of its opposite sides AB, DC are equal and parallel.

To prove that the other two sides AD, BC are also equal and parallel.

Join BD.

Proof. Then, because AB and DC are parallel, and BD meets them,

$\therefore$ the $\angle$ABD = the alt. $\angle$BDC.

Now in the $\triangle$s ABD, CDB, we have

(1) AB = CD,

(2) BD common,

and (3) the $\angle$ABD = the $\angle$CDB; (*proved above.*)

$\therefore$ the two $\triangle$s are congruent. (*Th. 4.*)

Hence AD = CB; ... ... (i)

and the $\angle$ADB = the $\angle$CBD.

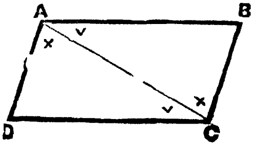
But these two $\angle$s are alt. $\angle$s;

$\therefore$ AD and BC are parallel. ... ... (ii)

Thus, from (i) and (ii), AD and BC are both equal and parallel.

Theorem 23. (Euc. I. 34.)

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram



Let ABCD be a parallelogram, of which AC is a diagonal.

To prove that

(i) AB = DC ;

(ii) AD = BC ;

(iii) the ∠ ABC = the ∠ ADC ;

(iv) the ∠ BAD = the ∠ BCD ;

(v) the △ ABC = the △ADC in area.

Proof. Because AB, DC are parallel, and AC meets them,

∴ the ∠ BAC = the alt. ∠ ACD.

Also, because AD, BC are parallel, and AC meets them,

∴ the ∠ DAC = the alt. ∠ ACB.

Hence, in the △s ABC, ADC, we have

(1) the ∠ BAC = the ∠ ACD, }
(2) the ∠ BCA = the ∠ CAD, } (*proved above.*)

and (3) AC common ;

∴ the two △s are congruent. (*Th. 18.*)

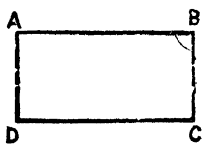
$$\left.\begin{array}{rl} \text{Hence} \quad AB &= DC, \\ AD &= BC, \\ \text{the } \angle ABC &= \text{the } \angle ADC, \end{array}\right\}$$

and the $\triangle$ ABC = the $\triangle$ ADC in area.

$$\left.\begin{array}{l} \text{Also, since the } \angle BAC = \text{the } \angle ACD, \\ \text{and the } \angle DAC = \text{the } \angle ACB\,; \end{array}\right\} \textit{(proved above.)}$$

$\therefore$ the whole $\angle$ BAD = the whole $\angle$ BCD. Q. E. D.

Cor. 1. ***A rectangle has all its angles right angles.***

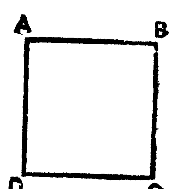


Let ABCD be a rectangle. Then by definition, (i) it is a parallelogram and (ii) it has one angle, say the $\angle$ B, a right angle.

Since AB and DC are ∥, the $\angle$ B + the $\angle$ C = 2 rt. $\angle$s; hence, the $\angle$ C *is a rt.* $\angle$.

Again, because ABCD is a par^{m}., the $\angle$ B = the $\angle$ D, and the $\angle$ C = the $\angle$ A; hence the $\angle$s A and D also are rt. $\angle$s.

Cor. 2. ***A square has all its sides equal and all its angles right angles.***



Let ABCD be a square; then by definition, (i) it is a rectangle and (ii) has two adjacent sides, say AB and BC, equal.

Since it is a rectangle, $\therefore$ *all its* $\angle$*s are rt.* $\angle$*s.*

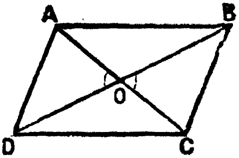
Again, since the square is also a par^{m};

$\therefore$ AB = DC and BC = AD.

But AB = BC, by hypothesis;

$\therefore$ *he s has also all its sides equal.*

Cor. 3. *The diagonals of a parallelogram bisect one another.*



Let the diagonals AC, BD of the par^m. ABCD intetsect at O.

To prove that AO = OC, **and** BO = OD.

In the $\triangle$s AOD, COB, we have

(1) the $\angle$ AOD = the $\angle$ COB,

(2) the $\angle$ OAD = the alt. $\angle$ OCB,

and (3) AD = BC; *(opp. sides)*

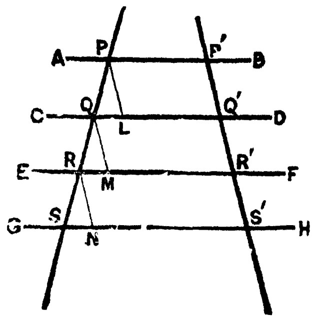
$\therefore$ the two $\triangle$s are congruent. *(Th. 18.)*

Hence AO = OC, and BO = OD.

Cor. 4. *A rhombus has all its sides equal.*

Theorem 24.

If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.



Let PS be a str. line which cuts the parallel str. lines AB, CD, EF, GH in P, Q, R, S so that PQ = QR = RS.

Let P'S' be any other str. line which cuts the parallels in P', Q', R', S'.

To prove that P'Q' = Q'R' = R'S'.

Suppose PL, QM, and RN are drawn ∥ to P'S', meeting CD in L, EF in M and GH in N.

Proof. Since CD is ∥ to EF, and PR meets them,

∴ the ∠PQL = the corresponding ∠QRM.

Also, since PL and QM are parallel, each being ∥ to P'R',

∴ the ∠QPL = the corresponding ∠RQM.

Then in the △s PQL and QRM, we have

(1) the ∠PQL = the ∠QRM,

(2) the ∠QPL = the ∠RQM,

and (3) PQ = QR ;

$$\left.\begin{array}{ll}\text{Hence} & PL = QM, \\ \text{By a similar mode of reasoning,} & QM = RN.\end{array}\right\}$$

Now, the quadrilateral PLQ′P′ is a par^{m}., since its opp. sides are ∥,

∴ $PL = P'Q'$. (*a*) (*Th. 23.*)

Similarly, the quadl. QMR′Q′ being a par^{m}.,

$QM = Q'R'$. (*b*) (*Th. 23.*)

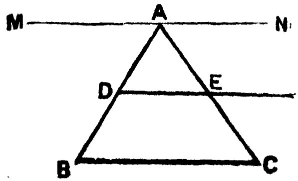
And the quadl. RNS′R′ being a par^{m}.,

$RN = R'S'$. (*c*) (*Th. 23.*)

Hence, from (*a*), (*b*) and (*c*), $P'Q' = Q'R' = R'S'$.

Q. E. D.

Cor. 1. *If* D *be the middle point of the side* AB *of a triangle* ABC, *then the straight line drawn through* D *parallel to* BC *will bisect* AC.



Let the str. line through D ∥ to BC intersect AC at E.

To prove that $AE = EC$.

If through A a str. line MN be supposed to be drawn ∥ to BC, then the parallels MN, DE, BC are cut by the two str. lines AB, AC.

Hence, the intercepts made on AB being equal, the corresponding intercepts on AC must be equal too.

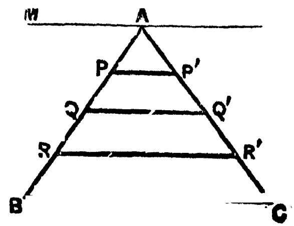
Hence $AE = EC$.

Cor. 2. *If* D *and* E *be the middle points of the sides* AB *and* AC *of a triangle* ABC, *then* DE *is parallel to* BC.

For, if DE be not ∥ to BC, let DE′ be ∥ to it, meeting AC in E′. Then E′ is the mid-point of AC, which is impossible. Hence DE is ∥ to BC.

Note. It is easy to see that BC = 2DE. For, if through E, EF be drawn parallel to AB to meet BC in F, then DEFB is a par^m., and $\therefore$ BF = DE. Also, since the Δs ADE and EFC are congruent, $\therefore$ DE = FC. Hence BF = FC, and BC = 2BF = 2DE.

Cor. 3. *If one side of a triangle be divided into any number equal parts, and if through the points of division, straight lines be drawn parallel to the base, then the points in which these parallels meet the other side will divide that side into the same number of equal parts.*



If the points P, Q, R divide the side AB of a triangle ABC into *four* equal parts; and if PP′, QQ′, RR′ drawn ∥ to BC meet AC in P′, Q′, R′: then AC also is divided into *four* equal parts at these points.

Supposing MAN to be drawn ∥ to BC, we find that the five parallels make equal intercepts on the line AB which cuts them.

Hence the intercepts made by them on another line AC, which cuts them, must also be equal.

EXERCISE (8).

1. ABCD is a parallelogram; if E and F are respectively the middle points of AB and CD, prove that the quadrilateral AECF is a parallelogram.

2. Prove that a quadrilateral is a parallelogram,

(i) if its opposite sides are equal;

or (ii) if its opposite angles are equal;

or (iii) if its diagonals bisect one another.

3. If the diagonals of a parallelogram are equal, prove that it is a rectangle.

4. From any two points on a straight line perpendiculars are drawn to a parallel straight line; prove that these perpendiculars are equal.

5. ABC and A′B′C′ are two triangles such that AB, AC are respectively equal and parallel to A′B′, A′C′. Prove that BC also is equal and parallel to B′C′.

6. Prove that the diagonals of a rhombus bisect each other at right angles.

7. If in a trapezium the non-parallel sides are equal, prove that the angles adjacent to either of the parallel sides are equal.

8. The straight line drawn through the middle point of a side of a triangle, parallel to the base, bisects the remaining side. Prove this without applying Theorem 21.

9. If D and E are respectively the middle points of the sides AB and AC of a triangle, prove that DE is parallel to BC, without employing the Indirect Method.

10. In example 9, prove that DE $= \frac{1}{2}$BC.

11. If the middle points of adjacent sides of any quadrilateral are joined, prove that the figure thus formed is a parallelogram.

12. If through the middle point of a side of a triangle a straight line be drawn parallel to the base, prove that it bisects any straight line drawn from the vertex to the base.

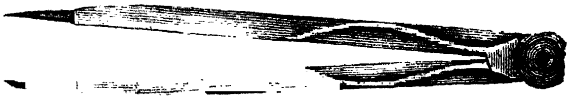
SECTION VI.

PROBLEMS.

In order that the constructions made may be *practically* as much accurate as possible, the student should provide himself with the following instruments :—

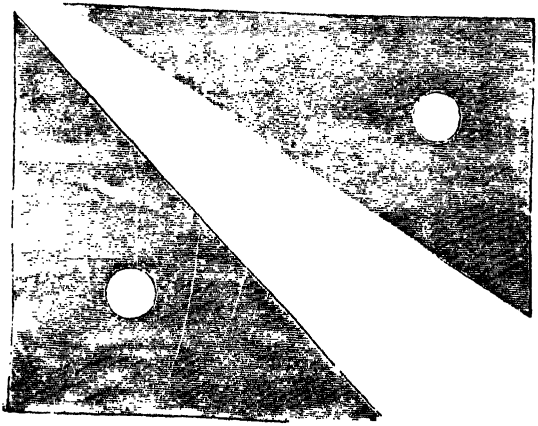
(1) **A Hard Pencil**; well sharpened, so that the traces made may be as fine as possible.

(2) **A Pair of Compasses** (also called Dividers).

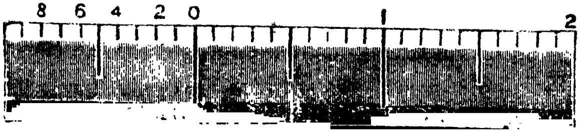


(3) **Pencil Compasses.**

(4) **Two Set-squares**; one with angles of 45° and the other with angles of 60° and 30°.



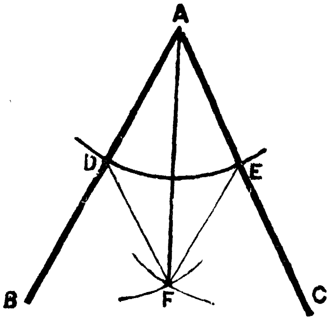
(5) A Graduated Flat Ruler; shewing inches and tenths of an inch on one side, and centimetres and tenths of a centimetre on the other.



Note. In the following Problems, however, the necessary constructions can be effected without the aid of a *graduated* ruler; simply a str. ruler will do. It is only in special cases, where the actual measurement of a line may be necessary, that a *graduated* ruler will be required.

Problem 1. (Euc. I. 9.)

To bisect a given angle.



Let BAC be the given angle.

It is required to divide it into two equal angles.

Cons. With centre A, and any radius, describe an arc of a circle cutting AB, AC at D and E respectively.

With centre D, and radius DE, describe an arc of a circle, on the side of DE remote from A ; also with centre E, and radius ED, describe another arc cutting the former at F.

Join AF.

Then AF is the bisector of the ∠ BAC.

Proof. Join DF, EF.

In the △s DAF, EAF, we have

(1) AD = AE, (radii of the same circle)
(2) DF = EF, (each being = DE)
and (3) AF common ;

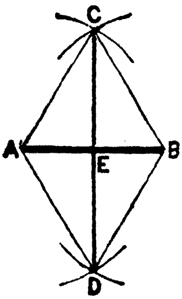
∴ the two △s are equal in all respects.

Hence the ∠ DAF = the ∠ EAF ;
that is, the ∠ BAC is bisected by AF. Q. E. F.

Note. The letter Q. E. F. at the end of a problem stand for *Quod Erat Faciendum*, that is, *which was to be done.*

Problem 2. (Euc. I. 10.)

To bisect a given finite straight line



Let AB be the given finite str. line.

It is required to divide it into two equal parts.

Cons. With centre A, and radius AB, describe two arcs, one on each side of AB.

With centre B, and radius BA, describe two other arcs cutting the former at C and D.

Join CD, cutting AB at E.

Then E is the middle point of AB.

Proof. Join AC, BC, AD, BD.

In the $\triangle$s ACD, BCD, we have

(1) AC = BC, (each being = AB)

(2) CD common,

and (3) AD = BD : (each being = AB)

$\therefore$ the two $\triangle$s are equal in all respects.

Hence the $\angle$ ACD = the $\angle$ BCD.

Again, in the $\triangle$s ACE, BCE, we have

(1) AC = BC,

(2) CE common,

and (3) the $\angle$ ACE = the $\angle$ BCE ; (*proved above*)

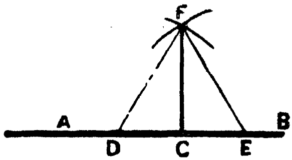
$\therefore$ the two $\triangle$s are equal in all respects.

Hence AE = EB ;

i.e., AB is bisected at E. Q. E. F.

Problem 3. (Euc. I. 11.)

To draw a straight line perpendicular to a given straight line from a given point in it.



Let AB be the given str. line, and C the given point in it.

It is required to draw from C a str. line $\perp$ to AB.

Cons. With centre C, and any convenient radius, describe a circle cutting AB at D and E.

With centres D and E, and radius DE, describe two arcs cutting each other at F.

Join CF.

Then CF is perpendicular to AB.

Proof. Join FD, FE.

In the $\triangle$s FDC, FEC, we have

(1) FD = FE, (each being = DE)

(2) DC = EC, (*Cons.*)

and (3) FC common ;

$\therefore$ the two $\triangle$s are equal in all respects.

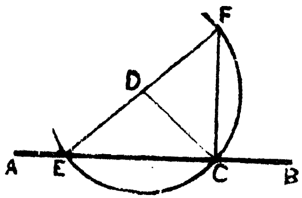
Hence the $\angle$ FCD = the $\angle$ FCE.

But these are adjacent angles ; and therefore each of them is a right angle.

$\therefore$ CF is $\perp$ to AB. Q. E. F.

Alternative Method.

Cons. Take any point D outside AB.



With centre D, and radius DC, describe a circle cutting AB at E.

Join ED, and produce it to meet the circle at F. Join CF.

Then CF is ⊥ to AB.

Proof. Join DC.

Since DE = DC, ∴ the ∠DCE = the ∠DEC.

Also ∴ DF = DC, ∴ the ∠DCF = the ∠DFC.

∴ the whole ∠ECF = the ∠FEC + the ∠EFC
= the ∠BCF.

And these are adjacent angles,

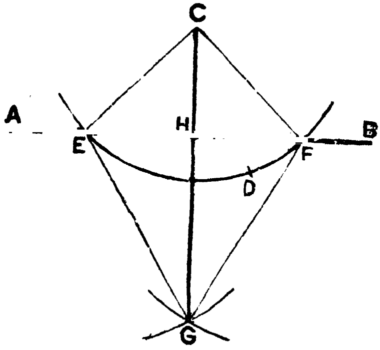
∴ CF is ⊥ to AB.

Note 1. That the angle **ECF** is a right angle may also be proved as follows: Since the ∠ **ECF** = the ∠ **FEC** + the ∠ **EFC**; ∴ each side of this equality = half the sum of the two sides = half of two rt. ∠s = one rt. angle.

Note 2. This method is applicable when the point **C** is near, or at one end of the straight line **AB**.

Problem 4. (EUC. I. 12.)

To draw a straight line perpendicular to a given straight line from a given point outside it.



Let AB be the given str. line, and C the given pt. outside it.

It is required to draw from C a straight line ⊥ to AB.

Cons. Take any point D on the side of AB remote from C.

With centre C, and radius CD, describe a circle cutting AB at E and F.

With centres E and F, and radius EF, describe two arcs, on the side of AB remote form C, cutting each other at G.

Join CG cutting AB at H.

Then CH is ⊥ to AB.

Proof. Join CE. CF. GE, GF.

In the △ s CEG, CFG. we have

(1) CE = CF,

(2) EG = FG, (each being = EF)

and (3) CG common;

∴ the two △s are equal in all respects.

Hence the $\angle$ECG = the $\angle$FCG.

Again in the $\triangle$s CEH, CFH, we have

(1) CE = CF,

(2) CH is common,

and (3) the $\angle$ECH = the $\angle$FCH :

$\therefore$ the two $\triangle$s are equal in all respects.

Hence the $\angle$CHE = the $\angle$CHF :

and these are adjacent angles. and $\therefore$ each of them is a rt. angle.

$\therefore$ CH is $\perp$ to AB. Q. E. F.

Alternative Method.

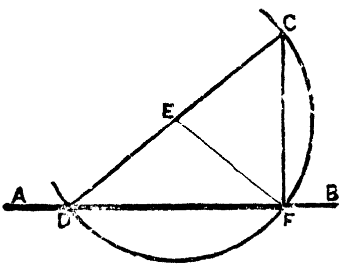
Cons. Take any point D in AB.

Join DC and bisect it at E.

With centre E, and radius EC, describe a circle cutting AB at D and F.

Join CF.

Then CF is $\perp$ to AB.



Proof. Join EF.

Since ED = EF, $\therefore$ the $\angle$EFD = the $\angle$EDF.

Also since EC = EF, $\therefore$ the $\angle$EFC = the $\angle$ECF.

$\therefore$ the whole $\angle$DFC = the $\angle$CDF + the $\angle$DCF

= the $\angle$BFC.

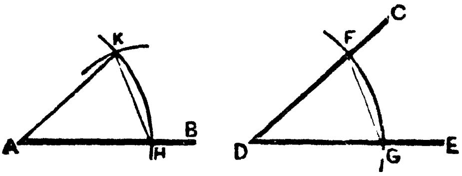
But these are adjacent $\angle$s, and $\therefore$ each of them is a right angle.

Hence CF is $\perp$ to AB. Q. E. F.

Note. This method is applicable when the point **C** is nearly opposite to one end of **AB**. The first method also may be applied by producing **AB** to any required length.

Problem 5. (Euc. I. 23.)

At a given point in a given straight line to make an angle equal to a given angle.



Let A be the given pt. in the given str. line AB, and CDE the given angle.

It is required to make at the pt. A in the str. line AB an ∠ equal to the ∠ CDE.

Cons. With centre D, and any radius, describe a circle cutting DC and DE at F and G respectively.

With centre A, and radius DG, describe an arc cutting AB at H; with centre H, and radius GF, describe an arc cutting the former at K.

Join AK.

Then KAH will be the reqd. angle.

Proof. Join FG, KH.

In the △s KAH, FDG, we have

(1) AH = DG, (*Cons.*)

(2) AK = DF, (*Cons.*)

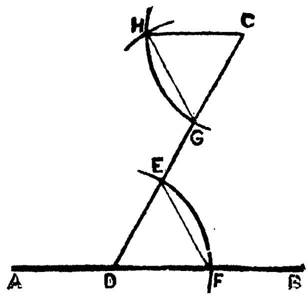
and (3) HK = GF; (*Cons.*)

∴ the two △s are equal in all respects.

Hence the ∠ KAH = the ∠ FDG. Q. E. F.

Problem 6. (Euc. I. 31.)

Through a given point to draw a straight line parallel to a given straight line.



Let C be the given point, and AB the given straight line.

It is required to draw through C a straight line parallel to AB.

Cons. Take any point D in AB, and join CD.

With centre D, and any radius, describe a circle cutting CD at E and DB at F.

With centre C, and radius DE, describe an arc, on the side of CD remote from B, cutting CD at G.

With centre G, and radius FE, describe an arc cutting the former at H.

Join CH.

Then CH is parallel to AB.

Proof. Join EF, HG.

In the $\triangle$s CHG, DFE, we have

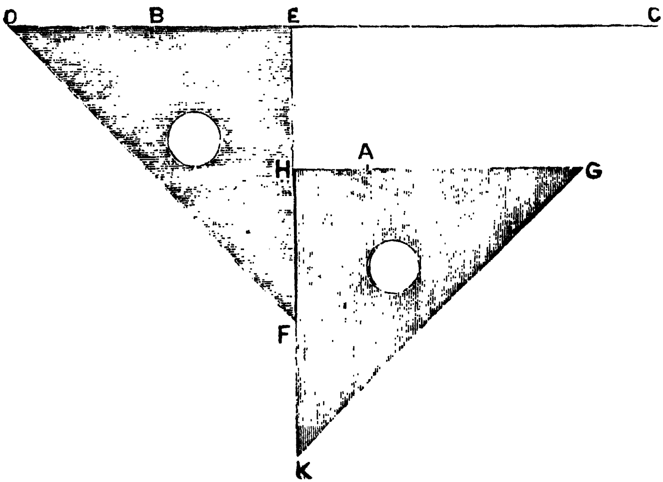
(1)	CH = DF,	(*Cons.*)
(2)	CG = DE,	(*Cons.*)
and (3)	GH = FE:	(*Cons.*)

Hence the ∠GCH = the ∠EDF, and these are alternate angles;

∴ CH is ∥ to AB. Q. E. F.

Alternative Method.

Let A be the given point, through which a str. line is to be drawn ∥ to the given str. line BC.



Cons. Place the Set-square DEF so that the edge DE may fall along BC.

Then slip the other Set-square GHK into the position shewn in the diagram, so that HG may pass through A.

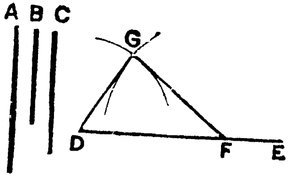
Now trace a line along HG.
Then GH is ∥ to BC.

Proof. The ∠DEF = the ∠GHE, each of them being a rt. ∠.

And these are alternate ∠s;
∴ HG is ∥ to BC. Q. E. F.

Problem 7. (Euc. I. 22.)

To construct a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be three given str. lines, of which A is the greatest, and any two of which are together greater than the third.

It is required to construct a triangle having its sides equal to the str. lines A, B, C.

Cons. Take any str. line DE, and from it cut off DF equal to A

With centre D, and radius B, describe a circle.

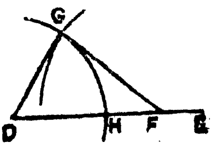
With centre F, and radius C, describe another circle cutting the former at G.

Join GD, GF.

Then GDF is the reqd. $\triangle$;

$$\left.\begin{array}{rr} \text{because, by construction,} & DF = A, \\ & DG = B, \\ \text{and} & FG = C. \end{array}\right\} \quad \text{Q. E. F.}$$

Note 1. If the circle described with centre **D**, and radius **B**, cuts **DF** at **H**, as in the following diagram, then **DH** = **B**. Now since

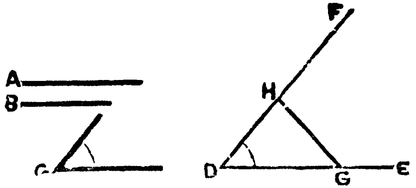


B and **C** are together greater than **DF**, ∴ **C** is greater than **FH**. Hence the circle described with centre **F**, and radius **C**, must evidently be partly inside and partly outside the former circle; which shews that the second circle must cut the first. If **B** and **C** were together not greater than **A**, then the two circles would not cut each other, and the construction would fail.

Note 2. The circles would also cut at a point on the other side of **DF**. Hence, with the given parts, *two* triangles can be constructed one on either side of **DF**.

Problem 8.

To construct a triangle having given two sides and the included angle.



Let A, B be two given str lines, and C a given angle.

It is required to construct a triangle having two of its sides equal to A and B, and having the angle included between these two sides equal to the ∠C.

Cons. Take any str. line DE.

At the point D in the str. line DE make the ∠EDF = the ∠C. (*Prob. 5.*)

From DE cut off DG = A, and from DF cut off DH = B.

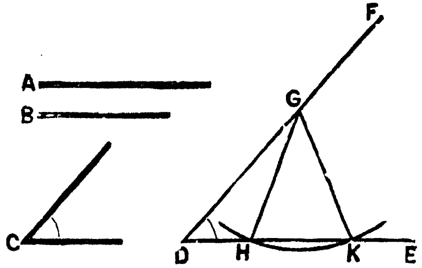
Join HG.

Then HDG is the △ required;

$$\left.\begin{array}{lrcl} \text{for} & DH & = & B, \\ & DG & = & A, \\ \text{and the} & \angle HDG & = & \text{the } \angle C. \end{array}\right\} \qquad \text{Q. E. F.}$$

Problem 9.

To construct a triangle having given two sides and the angle opposite to one of them.



Let A, B be two given str. lines, and C a given angle.

It is required to construct a △ having two of its sides equal to A and B, and having the ∠ opposite to the latter side = the ∠ C.

Cons. Take any str. line DE.

At the pt. D in the str. line DE, make the ∠ EDF = the ∠ C. (*Prob.* 5.)

From DF cut off DG = A.

With centre G, and radius B, describe an arc of a ⊙ cutting DE at H and K.

If H and K are on the same side of D, as in the above diagram, join both GH and GK.

Then evidently *each* of the △s GDH and GDK is the △ reqd. Q. E. F.

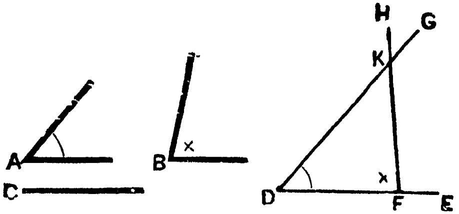
Note 1. There are two solutions to the problem, as above, only when **B** is less than **A** but greater than the perpendicular from **G** on **DE.** This is known as the **Ambiguous Case.**

Note 2. When **B** is less than the perpendicular from **G** on **DE,** the construction fails; or, in other words, the problem does not admit of a solution.

Note 3. If the given angle be a right angle, the problem amounts to constructing a right-angled triangle, of which the hypotenuse and one side are given.

Problem 10.

To construct a triangle having given two angles and the side adjacent to them.



Let A and B be two given ∠s, and C a given str. line.

It is reqd. to construct a △ having two of its ∠s equal to the ∠s A and B, and the side adjacent to these angles = the str. line C.

Cons. Take any str. line DE, and from it cut off DF = C.

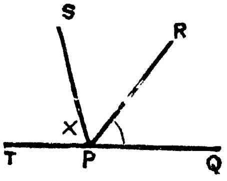
At the pt. D in the str. line DF, make the ∠FDG = the ∠A. (*Prob.* 5.)

At the pt. F in the str. line FD, make the ∠DFH = the ∠B. (*Prob.* 5.)

Let FH cut DG at K.

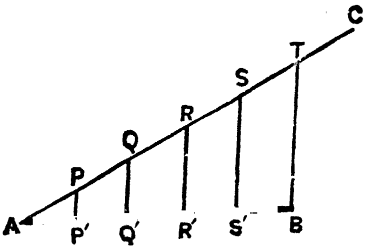
Then evidently, KDF is the △ reqd. Q. E. F.

Note. If the ∠DKF were given instead of the ∠DFK, the ∠DFK might be found as follows:—Take any str. line PQ; at the pt. P in the str. line PQ, make the ∠QPR = the ∠A; at the pt. P in the str. line PR, make the ∠RPS = the DKF; produce QP to any pt. T; then the ∠SPT = the ∠DFK, for each of them is supplementary to the sum of the ∠s A and DKF.



Problem 11.

To divide a given straight line into any number of equal parts.



Let AB be the given str. line, and suppose that it is required to divide it into *five* equal parts.

Cons. From A draw a str. line AC, of unlimited length, making any angle with AB.

From AC cut off successively five *equal* parts of *any* length, AP, PQ, QR, RS, ST.

Join TB; and through P, Q, R, S draw PP′, QQ′, RR′, SS′ each ‖ to TB, meeting AB in P′, Q′, R′, S′ respectively.

Then AB is divided into five equal parts at the points P′, Q′, R′, S′.

Proof. Since the side AT of the △ATB has been divided into five equal parts at the points P, Q, R, S, and through the points of division PP′, QQ′, RR′, SS′ have been drawn ‖ to the base TB;

∴ the points P′, Q′, R′, S′ also divide the other side AB into five equal parts. (*Th. 24, Cor. 3.*)

Q. E. F.

EXERCISE (9).

1. Take a straight line one inch in length, and construct an equilateral triangle on each side of it. Prove that the quadrilateral thus fromed is a *rhombus*.

2. ABC is a right angle. Through B draw a straight line BD within the angle so that the angle ABD may be equal to one-third of the angle ABC.

3. Trisect a right angle.

4. Take a straight line one and a half inches long, and on it construct an isosceles triangle having each of the two sides $2\frac{1}{4}$ inches in length.

5. AB is a given straight line. Shew how to draw a straight line AC so that the angle BAC may be equal to 30°.

6. On a given straight line as hypotenuse, construct a right-angled triangle of which the acute angles are 30° and 60°.

7. Construct a right-angled triangle having given the hypotenuse and one side.

8. Construct an isosceles triangle having given the vertical angle and the length of the perpendicular from the vertex to the base.

9. From a given point outside a given straight line, draw a line making an angle with the given line equal to a given angle.

10. Construct a triangle of which the base angles shall be equal to two given angles, and the perpendicular from the vertex to the base equal to a given straight line.

11. Construct a triangle of which the two sides and the perpendicular from the vertex to the base are given.

12. Construct an equilateral triangle having given the perpendicular from the vertex to the base.

13. Construct a triangle having given the base, one of the angles at the base, and the sum of the sides.

14. Two sides of a triangle are respectively 3 inches and 2 inches in length, and the angle opposite to the latter side is equal to 30°; construct the triangle.

Also construct the triangle when the length of the latter side is $3\frac{1}{2}$ inches.

Shew that in the former case there are two solutions, whilst in the latter there is only one.

15. If the side BA of a triangle ABC be produced to D so that AD = AC, and if DC be joined, prove that the angle BDC is half of the angle BAC. Hence shew how to construct a triangle having given the base, the vertical angle, and the sum of the sides.

16. ABC is a triangle. If D be taken on CB produced such that BD = AB, and if E be taken on BC produced such that CE = AC, and if AD and AE be joined; prove that the angle ADE is half of the angle ABC, and the angle AED is half of the angle ABC. Hence shew how to construct a triangle having given the *perimeter* and the base angles. (The sum of the three sides of a triangle is called its **perimeter.**)

17. Take a straight line 4 inches long, and divide it into seven equal parts.

SECTION VII.

LOCI.

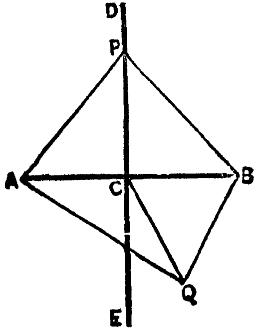
When a point moves and its motion is restricted by some given condition, the path traced out by the point is called its **locus**.

For instance, when a point moves so that its distance from a fixed point is always the same, its locus is evidently the circumference of a circle of which the fixed point is the centre and the given distance, the radius.

Note. When any number of points, each of which satisfies a given condition, lie on a particular line, that line is also called the locus of those *points*. From this point of view, the locus of points, each of which lies at a given distance from a given point, is the circumference of the circle which is described with the given point as centre and the given distance as radius.

Theorem 25.

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.



Let A and B be two fixed points, let C be the mid-point of the str. line AB.

Let DE be the unlimited str. line which passes through C and is perpendicular to AB.

To prove that if a point moves so that its distances from the points A and B are always equal, the only path along which it can move is the str. line DE.

Proof. (i) Take any pt. P on the str. line DE.

Join PA, PB.

In the $\triangle$s PCA, PCB, we have

(1) AC = CB,

(2) PC common,

and (3) the $\angle$ PCA = the $\angle$ PCB ;

$\therefore$ the two $\triangle$s are equal in all respects.

Hence PA = PB.

Thus, *every point* on the str. line DE is equidistant from A and B ... ... (α)

(ii) Again, it may be proved that no pt. outside the str. line DE is equidistant from A and B.

For, if possible, let the pt. Q, which is *not* on DE, be equidistant from A and B.

Join QA, QB, QC.

Then in the $\triangle$s QCA, QCB, we have

(1) QA = QB,

(2) CA = CB,

and (3) QC common ;

$\therefore$ these two $\triangle$s are equal in all respects.

Hence the $\angle$ QCA = the $\angle$ QCB ;

and $\therefore$ QC is $\perp$ to AB, which is impossible.

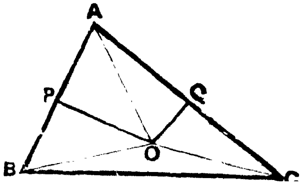
Thus *no point outside* DE *can be equidistant from* A *and* B . ... (β)

From (α) and (β) we see that *every pt.* on the str. line DE, *and no other pt.*, is equidistant from A and B.

Hence, the only path that the moving point can describe is the unlimited str. line DE. Q. E. D.

Cor. *The point of intersection of the perpendicular bisectors of the straigot lines* AB *and* AC *is equidistant from the points* A, B, C.

Let PO and QO, which meet at O, be the ⊥ bisectors of AB and AC respectively.

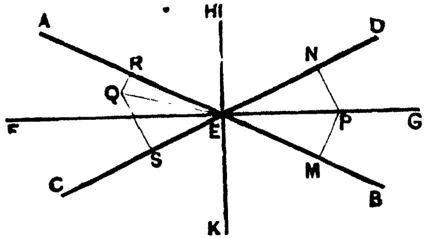


Now, O being a pt. on PO, OA = OB.
Also, O being a pt. on QO, OA = OC.
Hence OA, OB, OC are equal to one another.

Note The perpendicular bisector of the straight line **AB** may also be said to be the locus of *points* which are equidistant from **A** and **B**.

Theorem 26.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.



Let AB, CD be two given str. lines which intersect at E.

Let FG be the bisector of the ∠s AEC and BED ; and let HK be the bisector of the ∠s AED and BEC.

To prove that a point which moves so that its distances from the str. lines AB and CD are always equal, must move along one or other of the two lines FG and HK.

Proof. (i) Take *any* pt. P on the line FG.

From P draw PM and PN ⊥ to AB and CD respectively.

In the △s PEM, PEN, we have

(1) the ∠PEM = the ∠PEN,

(2) the ∠PME = the ∠PNE, (*Rt.* ∠*s.*)

and (3) the side EP common ;

∴ the two △s are equal in all respects.

Hence PM = PN.

Thus, *every pt.* on the line FG is equidistant from the lines AB and CD ;

and similarly *every pt.* on the line HK is equidistant from AB and CD ; ... ... (α)

(ii) Again, it may be proved that no pt. which lies outside FG and HK, is equidistant from AB and CD.

For, if possible, let the pt. Q, which is *not* on FG or HK, be equidistant from AB and CD.

From Q draw QR, QS ⊥ to AB and CD ; and join QE.

Then, by hypothesis, QR = QS.

Now, in the *right-angled* △s QRE, QSE, we have

(1) the hypotenuse QE common,

and (2) QR = QS ;

∴ the two △s are equal in all respects. (*Th. 19.*)

Hence, the ∠QER = the ∠QES ;

i.e., QE is the bisector of the ∠AEC, which is impossible.

Thus, *no* pt. which lies outside the lines FG and HK, can be equidistant from AB and CD ... ... (β)

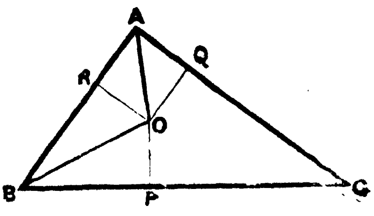
From (α) and (β) we see that *every* pt. on the lines FG and HK, *and no other pt.*, is equidistant from AB and CD.

Hence the lines FG and HK form the locus of a point which moves under the proposed condition.

Q. E. D.

Cor *The point of intersection of the bisectors of the angles* A *and* B *of a triangle* ABC *is equidistant from the sides of the triangle.*

Let AO and BO, which meet at O, be the bisectors of the ∠s A and B.



From O draw OP, OQ, OR ⊥s to BC, CA, AB respectively.

Since O is on the bisector of the $\angle$ A, $\therefore$ OR = OQ.
Also, since O is on the bisoctor of the $\angle$ B, $\therefore$ OR = OP.

Hence, OP, OQ, OR are equal to one another ;
i.e., the pt. O is equidistant from the sides BC, CA, AB.

Note. The lines **FG** and **HK** may also be said to be the locus of *points* which are equidistant from **AB** and **CD.**

EXERCISE (10).

1. Prove that the locus of a point which moves in such a manner that its distance from a given straight line is always the same, consists of two siraight lines parallel to the given line, one on either side of it.

2. What is the locus of points which are equidistant from two given points ? Find a point in a given straight line so that its distances from two given points outside the line may be equal.

3. What is the locus of points which are equally distant from two intersecting straight lines ? Find a point in a given straight line so that it may be equally distant from two intersecting straight lines.

4. Find a point which is equally distant from two fixed points and is also at a given distance from a third fixed point. When is this problem impossible ?

5. Find a point which is at a distance of 2 inches from one fixed point and 3 inches from another. When is this problem impossible ?

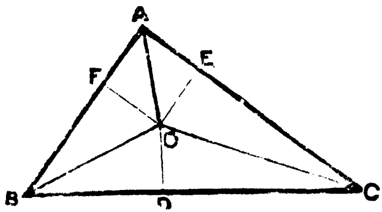
6. AB and AC are two fixed and unlimited straight lines. Find a point which is at a distance of 2 inches from AB and 3 inches from AC.

7. Find a point in a given straight line at a given distance from another given straight line.

SECTION VIII.

MISCELLANEOUS PROPOSITIONS.

1. *The bisectors of the angles of a triangle meet at a point.*



Let ABC be a $\triangle$. and let the bisectors of the $\angle$s B and C meet at O.

Join AO.

To prove that AO is the bisector of the $\angle$ A.

From O draw OD, OE, OF $\perp$ to BC, CA, AB respectively.

Proof. Because BO bisects the $\angle$ ABC,

$\therefore$ any pt. in BO is equidistant from BC and BA.

(*Th.* 26.)

$\therefore$ OD = OF.

For a similar reason, any pt. in CO is equidistant from CB and CA ;

$\therefore$ OD = OE.

Hence OF = OE.

$\therefore$ O is on the bisector of the $\angle$ BAC ; (*Th.* 26.)

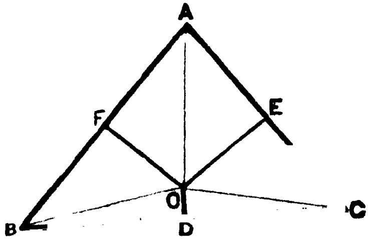
i.e., AO is on the bisector of the $\angle$ BAC.

Thus, the bisectors of the $\angle$s A, B, C meet at a pt.

Q. E. D.

Note. When three or more straight lines meet at a point, they are said to be **Concurrent**. So the above proposition might be stated as follows :—*The bisectors of the angles of a triangle are concurrent.*

2. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*



Let ABC be a $\triangle$, and D, E, F be the mid-points of BC, CA, AB respectively.

From F and E draw $\perp$s to AB, AC meeting at O.

Join OD.

To prove that OD is $\perp$ to BC.

Join OA, OB, OC.

Proof. Because O is on the perpendicular bisector of AB,

$\therefore$ OA = OB. (*Th.* 25.)

For a similar reason,

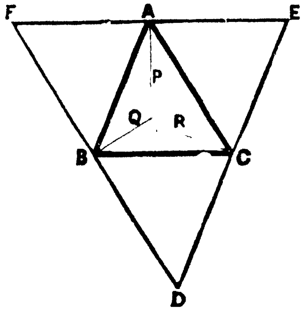
OA = OC.

Hence OB = OC ;

and $\therefore$ O is on the $\perp$ bisector of BC. (*Th.* 25.)

Hence OD is $\perp$ to BC. *Q. E. D.*

3. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



Let ABC be a △, and let AP. BQ. CR, when produced, be ⊥ to BC, CA, AB respectively.

To prove that AP, BQ, CR **(when produced) are concurrent.**

Through A, B and C, draw str. lines ∥ to BC, CA and AB respectively, thus forming the △ DEF.

Proof. Since FE is ∥ to BC. and FD is ∥ to AC,

∴ AFBC is a par^{m}.

∴ FA = BC.

Similarly, AE = BC.

Hence, FA = AE; and ∴ A is the mid-point of EF.

Similarly, B is the mid-pt. of FD, and C is the mid-pt. of DE.

Now, since BC is ∥ to FE, and AP produced is ⊥ to BC,

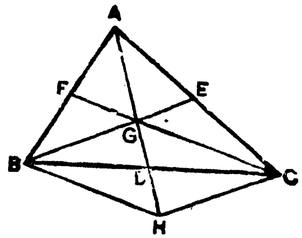
∴ AP is also ⊥ to FE.

Similarly, BQ is ⊥ to FD and CR is ⊥ to DE.

Thus, AP, BQ and CR are ⊥s to the sides of the △ EFD at their middle points; and ∴ they are concurrent. (*Last Prop.*)

Q. E. D.

4. *The medians of a triangle are concurrent; the common point dividing each median into two parts, of which the part nearer the vertex is double the other.*



Let E, F be the mid-pts. of the sides CA and AB of the △ ABC, and let the medians BE, CF intersect at G.

Join AG, and produce it to meet BC at D.

To prove that D is the mid-pt. of BC.

Through B draw BH ∥ to FC, and produce AD to meet BH at H.

Join CH.

Proof. (1). In the △ ABH, F is the mid-pt. of AB, and FG is ∥ to BH ;

∴ G is the mid-pt. of AH. (*Th. 24, Cor. 1.*)

Again, in the △ AHC, G and E are the mid-pts. of the sides AE and AC ;

∴ GE is ∥ to HC. (*Th. 24, Cor. 2.*)

Thus the opp. sides of the quadl. BGCH are ∥, and ∴ is a par^m.

But the diagonals of a par^m. bisect one another ;

∴ D is the mid-pt. of BC.

Thus the three medians AD, BE, CF meet at the point G. Q. E. D.

(2) Since D is the mid-pt. of GH,

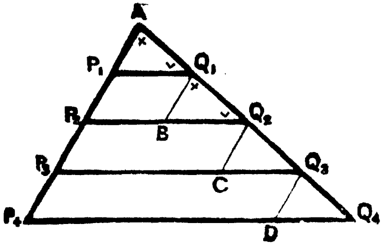
∴ GH = 2GD.

But $AG = GH$, $\therefore AG = 2GD$.
Also, $BG = HC = 2GE$; and, $CG = HB = 2GF$; } (*Th.* 24, *Cor.* 2, *Note.*)

Q. E. D.

Note.. The point of intersection of the medians of a triangle is called the **Centroid** of the triangle. Thus, in the above figure, the point **G** is the *Centroid* of the $\triangle ABC$.

5. AP_1Q_1 *is a triangle ; from* AP_1 *produced,* P_1P_2, P_2P_3, P_3P_4 *are cut off each equal to* AP_1 ; P_2Q_2, P_3Q_3, P_4Q_4 *are drawn* ∥ *to* P_1Q_1, *meeting* AQ_1 *produced in* Q_2, Q_3, Q_4. *Prove that* $P_2Q_2 = 2P_1Q_1$, $P_3Q_3 = 3P_1Q_1$, $P_4Q_4 = 4P_1Q_1$.



Draw Q_1B, Q_2C, Q_3D each ∥ to AP_4, meeting P_2Q_2, P_3Q_3, P_4Q_4 in B, C, D respectively.

Proof. AQ_1, Q_1Q_2, Q_2Q_3, Q_3Q_4 are equal to one another. (*Th.* 24, *Cor.* 3.)

Now, since the $\triangle$s AP_1Q_1, Q_1BQ_2 are congruent,

$\therefore \quad P_1Q_1 = BQ_2 \quad \ldots \quad \ldots \quad \ldots \quad (\alpha)$

Similarly, $BQ_2 = CQ_3$, and $CQ_3 = DQ_4$.

Thus, P_1Q_1, BQ_2, CQ_3, DQ_4 are equal to one another.

Again, since the quadl. $P_1P_2BQ_1$ is a par^{m}.,

$\therefore \quad P_1Q_1 = P_2B \quad \ldots \quad \ldots \quad \ldots \quad (\beta)$

Similarly, $P_2Q_2 = P_3C$ and $P_3Q_3 = P_4D$.

(1) From (α) and (β), $P_2Q_2 = 2P_1Q_1$.

(2) $P_3Q_3 = P_3C + CQ_3 = P_2Q_2 + P_1Q_1' = 3P_1Q_1$.

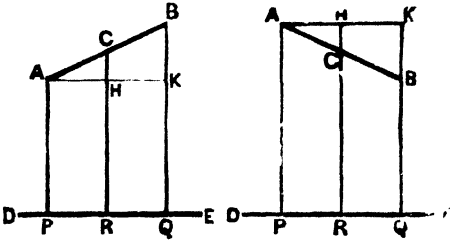
(3) $P_4Q_4 = P_4D + DQ_4 = P_3Q_3 + P_1Q_1 = 4P_1Q_1$.

Q. E. D.

Note. Hence, if **DE** be drawn parallel to the base **BC** of a triangle **ABC**, meeting **AB**, **AC** in **D** and **E**, the following conclusions are obvious :— (1) if **AB** = 2**AD**, then **BC** = 2**DE** ;
(2) if **AB** = 3**AD**, then **BC** = 3**DE** ;
(3) if **AB** = 4**AD**, then **BC** = 4**DE** ; and so on.

6. *C is a point in a finite straight line* AB ; DE *is an unlimited straight line such that the points* A *and* B *are on one side of it ;* AP, BQ, CR *are perpendiculars to* DE. *Prove that* (1) *if* C *is the middle point of* AB, *then* 2CR = AP + BQ ; (2) *if* BC = 2AC, *then* 3CR = 2AP + BQ.

(1) Let C be the middle point of AB.



Through A draw AK ∥ to DE meeting RC, QB, or those lines produced, in H and K.

The quadls. APRH and HRQK are par^{ms}. ;

∴ AP = HR = KQ.

Again, since AB = 2AC, and CH is ∥ to BK,

BK = 2CH. (*Last Prop., Note.*)

Hence, in fig. (1),

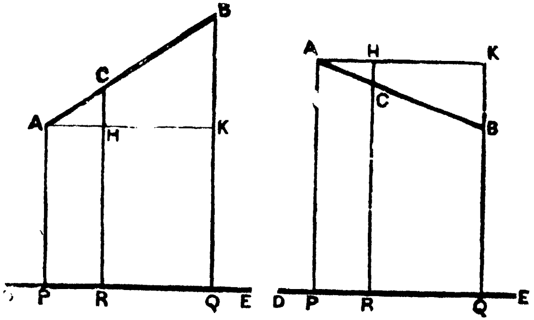
2CR = 2HR + 2CH

= AP + KQ + BK = AP + BQ.

and in fig, (2),

$$2CR = 2HR - 2CH$$
$$= AP + KQ - BK = AP + BQ.$$

(2) Let BC be $= 2AC$.



(1) (2)

Through A draw AK ∥ to DE, meeting RC, QB, or those lines produced, in H and K.

The quadls. APRH and HRQK are par^{ms};

$$\therefore \quad AP = HR = KQ.$$

Again, since $AB = 3AC$, and CH is ∥ to BK,

$$\therefore \quad BK = 3CH. \qquad \textit{(Last Prop., Note.)}$$

Hence in fig. (1),

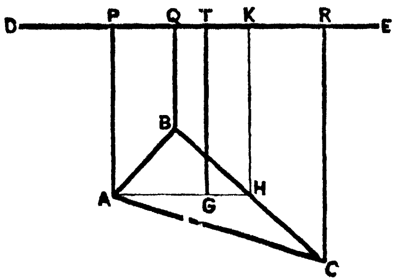
$$3CR = 3HR + 3CH$$
$$= 2AP + KQ + BK = 2AP + BQ.$$

And in fig. (2),

$$3CR = 3HR - 3CH$$
$$= 2AP + KQ - BK = 2AP + BQ.$$

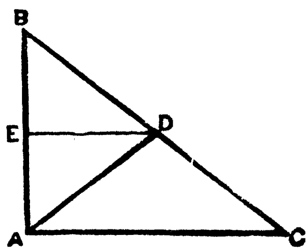
Q. E. D.

Note. Hence, if **G** be the *centroid* of a triangle **ABC**, and if **AP**, **BQ**, **CR**, **GT** be perpendiculars to an unlimited straight line, **DE**, which is such that the points **A**, **B**, **C** are all on one side of it then $3GT = AP + BQ + CR$.



For, let **H** be the mid-pt. of **BC**, and draw **HK** $\perp$ to **DE** ; then **G** is on **AH**, so that **AG** = 2**GH**. Hence 3**GT** = **AP**+2**HK**=**AP**+**BQ**+**CR**.

7. *If* D *be the middle point of the hypotenuse* BC *of a right-angled triangle* ABC, *prove that* DA = $\frac{1}{2}$BC.



Through D draw DE $\perp$ to AB ; then, since DE and CA are both $\perp$ to AB, $\therefore$ DE is $\parallel$ to CA.

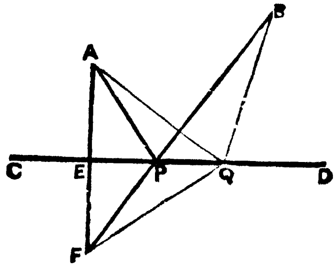
Hence, since D is the mid-pt. of BC, E also is the mid-pt. of BA ; (*Th. 24, Cor. 1.*)

$\therefore$ ED is the $\perp$ bisector of BA.

Hence, DB = DA.

But DB = $\frac{1}{2}$BC ; $\therefore$ DA = $\frac{1}{2}$BC. Q. E. D.

8. *A and B are two points on the same side of an unlimited straight line CD. P is a point in CD such that the angles APC and BPD are equal. Prove that, of all points on the straight line CD, P is that the sum of whose distances from A and B is the least.*



Let Q be *any other* pt. in CD.

Join AQ, BQ.

To prove that AQ + QB > AP + PB.

Produce BP to F, making PF = PA; join AF, cutting CD in E.

Join FQ.

Proof. In the $\triangle$s APE, FPE, we have

(1) AP = PF,

(2) PE common,

and (3) the $\angle$ APE = the $\angle$ EPF

($\because$ each of them = the $\angle$ BPD),

$\therefore$ the two $\triangle$s are congruent.

Hence AE = EF, and EP is $\perp$ to AF.

Thus, CD is the $\perp$ bisector of the str. line AF.

Hence, QA = QF;

and $\therefore$ AQ + QB = FQ + QB.

Also AP + PB = FB. (*Cons.*)

But FQ + QB is > FB,

$\therefore$ AQ + QB is > AP + PB. Q. E. D.

Note. It is easy to see that there is only *one* point in **CD** such that the str. lines drawn from it to **A** and **B** are *equally* inclined to **CD**.

EXERCISE (11).

1. ABC is a triangle of which the side AC is greater than the side AB. AD is drawn bisecting the angle BAC, and meeting BC in D. Prove that CD is greater than BD.

2. If D be the middle point of the base BC of a triangle ABC, and if AD be joined, prove that AB + AC $>$2AD.

3. Through a given point draw a straight line such that the perpendiculars on it from two fixed points may be on opposite side of it and equal to each other.

4. ABCD is a parallelogram ; E and F are the mid-points of AB and CD respectively. If DE, BF intersect the diagonal AC at G and H, prove that AG = GH = HC.

5. Prove that the medians of a triangle meet at a point which is a point of trisection on each of them. Hence find a method of trisecting a given straight line.

6. Construct a triangle having given the base, the perpendicular from the vertex to the base, and the line joining the vertex to the mid-point of the base.

7. ABC is a triangle. If AB, AC be produced to D and E respectively, prove that the bisectors of the angles DBC, ECB meet at a point which lies on the bisector of the angle BAC.

8. Through a given point draw a straight line making with a fixed straight line an angle equal to a given angle. Hence show how to construct a triangle of which the base angles and the perpendicular from the vertex to the base are given.

9. Construct a right-angled triangle having given the hypotenuse and the sum of the sides.

10. Prove that the locus of the vertices of all right-angled triangles which have a common hypotenuse is a circle.

11. Construct an isosceles triangle having given the base, and the sum of one of the equal sides and the perpendicular from the vertex to the base.

12. AB, AC are two given straight lines; through a given pt. D between them, draw a straight line meeting AB, AC in G and H, so that GH may be bisected at D.

13. OA, OB, OC are three given straight lines, of which OB lies between OA and OC. Draw a straight line terminated by OA and OC, and bisected by OB.

14. ABCD is a parallelogram, and MN is an unlimited straight line outside it. If AP, BQ, CR, DS be perpendiculars to MN, prove that AP + CR = BQ + DS.

15. If C is the middle point of a straight line AB, and D is any point in CB, prove that AD − DB = 2CD.

Hence prove that if C is the mid-point of a given straight line AB, where the points A and B are on opposite sides of an unlimited straight line DE, and if AP, BQ, CR be perpendiculars to DE, and AP > BQ, then AP − BQ = 2CR.

16. Construct a triangle, having given the base, an angle at the base, and the difference of the sides.

17. BC is the base of an isosceles triangle ABC, EAD is drawn parallel to BC, and *any* point P is taken on AD. Prove that BP + PC > BA + AC.

Prove also that if Q be *any* point above the line DE, $BQ + QC > BA + AC$.

Hence prove that if R be *any* point *above* BC, such that $BR + RC = BA + AC$, then R must be *below* DE.

18. Construct a triangle, having given the middle points of the sides.

19. Of all triangles which stand on the same base and which have their vertices in a straight line parallel to the base, prove that the triangle which is isosceles has the *least* perimeter.

20. Prove that any side of a triangle is greater than the difference of the other two sides. Hence prove that if P is a point in an unlimited straight line MN, such that PA and PB are equally inclined to it, where A and B are two fixed points on opposite sides of MN, then of all points in the given straight line, P is that the *difference* of whose distances from A and B is the *greatest*.

BOOK II

Areas.

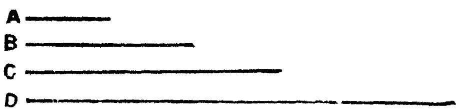
SECTION I.

Fundamental Ideas and Definitions.

1. If two straight lines A and B be such that B can be divided into a number of parts each of which is equal to A then the line A is said to be a **measure** of the line B.

Note 1. When B is divisible into parts each of which is equal to A it may be said that B *contains* A *an exact number of times.* Hence, we may say that *one straight line is a measure of another when the latter contains the former an exact number of times.*

2. If two or more straight lines, B, C, D be such that *each* of them contains a given straight line A an exact number of times, then A is said to be a **common measure** of B, C, D.



Note 2. In the above diagram A is contained twice in B, three times in C, and five times in D; *i.e.*, an *exact number* of times in each. Hence A is a *common measure* of B, C, D.

3. Sizes, or lengths of different straight lines are ascertained by finding how often each of them contains any smaller straight line arbitrarily chosen.

Note 3. *To measure a straight line* is to ascertain its length in the manner above indicated.

4. Any convenient straight line that is arbitrarily chosen for the purpose of measuring other straight lines, is called the **unit of length.**

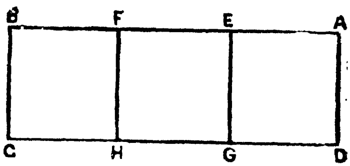
Note 1. The *number* of times that any given straight line contains the unit of length is called the **numerical measure** of the straight line, and the given line is said to be *represented* by that number.

Note 2. When we say that one inch is the unit of length, we mean that *a straight line whose length is one inch* has been taken as the unit.

5. The amount of surface enclosed by the bounding lines of a figure is called its **area.**

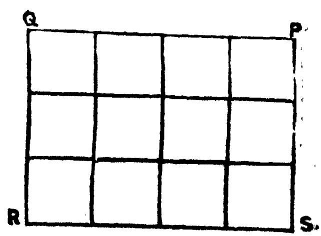
6. The **unit of area** is the area of a square of which a side is equal to the unit of length.

Note 1. Let **ABCD** be a rectangle of which the side **BC** = the unit of length, and the side **AB** = three times the unit. Then the area of the rectangle is equal to *three* units of area. For, if **AB** be divided into three equal parts at **E** and **F**, and if through the points of division, **EG** and **FH** be drawn ∥ to **BC**, then evidently each of the figures **BH**, **FG** and **ED** is a square of which a side = the unit of length.



Note 2. *If* **PQRS** *be a rectangle of which the side* **PQ** = *4 units of length, and the side* **QR** = *3 units of length, then the area of the rectangle is equal to 12 (i.e. 3×4) units of area.*

For, if **PQ** be divided into 4 equal parts, and **QR** into 3 equal parts, and if through the pts. of division in either side, lines be drawn parallel to the other; then the whole rectangle is divided into 3 rows of squares, a side of each square being equal to the unit of length. There are 4 squares in each row; the total number of squares = 3 times 4 = 12, and hence the area of the rectangle = 12 units of area.



Note 3. Hence, generally speaking, if the adjacent sides of a rectangle be equal to a and b units of length respectively (a and b being integers), then the area of the rectangle $= ab$ units of area; and if a side of a square be equal to a units of length, the area of the square $= a^2$ units of area.

Note 4. Of two adjacent sides of a rectangle, the length of the greater side is called the *length of the rectangle*, and the length of the smaller side is called its ***breadth***.

7. A **Square inch** is the *area* of a square of which a side is one inch in length.

Note 1. There is a clear distinction between "two square inches" and "two inches square." The former means an area twice as large as the area of a square of which a side is one inch, whilst the latter means a *square* of which a side is two inches in length. In fact, the area of a "two inches square" is *four square inches*, as can be easily seen by a diagram.

Note 2. *If* **ABCD** *be a rectangle of which the sides* **AB**, **BC** *are respectively* $4\frac{2}{3}$ *inches and* $3\frac{1}{5}$ *inches in length, then its area is equal to* $(4\frac{2}{3} \times 3\frac{1}{5})$ *square inches.*

For, taking $\frac{1}{15}$ of an inch as the unit of length, **AB** and **BC** are found to contain 70 and 48 units respectively; and ∴ the required area $= 70 \times 48$ *units of area*. But as 15 units of length = one inch, one sq. inch $= 15 \times 15$ *units of area*. Hence, *the number of sq. inches* in the area of the rectangle $= \dfrac{70 \times 48}{15 \times 15} = \frac{14}{3} \times \frac{16}{5} = 4\frac{2}{3} \times 3\frac{1}{5}$, which proves the proposition.

8. The perpendicular drawn from the vertex of a triangle to the base, or the base produced, is called the **altitude**, or **height**, of the triangle.

Note 1. Triangles are said to be *between the same parallels* when their bases lie in one straight line and the vertices in another which is parallel to the former.

Note 2. It may be easily shewn that if two triangles are on the same base and between the same parallels, their altitudes are equal.

Note 3. Triangles having *equal* altitudes are said to have the *same altitude*.

9. The perpendicular drawn to the base of a parallelogram from *any* point in the opposite side is called the **altitude**, or **height**, of the parallelogram.

Note 1. Parallelograms are said to be *between the same parallels* when their bases lie in one straight line, and the sides opposite to the bases in another which is parallel to the former.

Note 2. It may be easily shewn that if two parallelograms are on the same base and between the same parallels their altitudes are equal

Nate 3. Parallelograms having *equal* altitudes are said to have the *same altitude.*

10. A rectangle of which two adjacent sides are AB and BC, is said to be **the rectangle contained by AB and BC.**

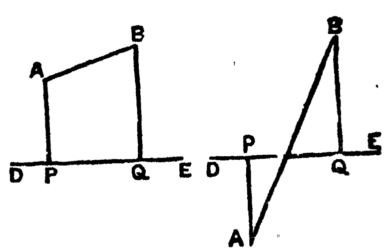
Note 1. A rectangle of which two adjacent sides are equal to two given straight lines **P** and **Q**, is also said to be **the rectangle contained by P and Q.**

Note 2. "The rectangle contained by **AB** and **BC**" is briefly written as " the rect. **AB**, **BC** ;" or more briefly, as "**AB, BC**".

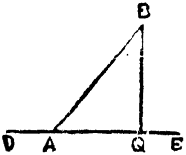
11. A straight line AB is said to be divided **internally** or **externally**, at any point D, according as the point D is taken on AB, or on AB produced. In either case, AD and DB are called the *segments* of the line AB.

Note. In the case of external division, **AD** and **DB** are regarded as the segments of the line **AB** only in the sense that they are the distances of **D** from **A** and **B** respectively. In the case of internal division, **AD** and **DB** are not only the distances of **D** from **A** and **B** but they also together make up **AB**.

12. If from the extremities of a straight line AB, perpendiculars AP and BQ be drawn to an unlimited straight line DE, then PQ is said to be the **Projection** of AB on DE.



Note. If **A** is on the straight line **DE**, then **AQ** becomes the projection.



EXERCISE (12).

1. Shew that if one side of a rectangle be a measure of another, the figure can be divided into a nnmber of equal squares.

2. Shew that if two adjacent sides of a rectangle have a common measure, the figure can be divided into a number of equal squares, each square having its sides equal to the common measure.

3. If the unit of length be 3 inches, shew that the unit of area is divisible into 9 one-inch squares.

4. If the length and breadth of a rectangle are 12 and 8 inches respectively, shew that the figure can be divided into 6 four-inch squares.

5. If two adjacent sides of a rectangle are $2\frac{1}{2}$ and $3\frac{1}{3}$ inches respectively, prove that its area $= (2\frac{1}{2} \times 3\frac{1}{3})$ square inches.

6. If two adjacent sides of a rectangle are 3·4 and 5·7 inches respectively, prove that its area $= (3{\cdot}4 \times 5{\cdot}7)$ square inches.

7. If two adjacent sides of a rectangle are 16 and 12 inches respectively, what is the *smallest* number of squares into which the figure can be divided ?

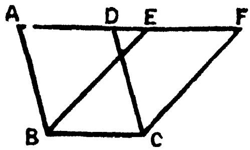
8. If the unit of length is doubled, the unit of area is quadrupled ; if the unit of length is trebled, the unit of area is increased nine times. Illustrate this statement.

SECTION II.

THEOREMS.

Theorem 1. (EUC. I. 35.)

Parallelograms on the same base and between the same parallels are equal in area.



Let the par^{ms}. ABCD and EBCF be on the same base BC and between the same parallels AF, BC.

To prove that the par^{ms}. ABCD and EBCF are equal in area.

Proof. Since DC is ‖ to AB,

∴ the ∠FDC = the int. opp. ∠EAB.

Again, since EB is ‖ to FC,

∴ the ∠DFC = the ext. ∠AEB.

Thus, in the two △s FDC, EAB, we have

(1) the ∠FDC = the ∠EAB,

(2) the ∠DFC = the ∠AEB,

and (3) DC = AB ; (*opp. sides of a par^m.*)

∴ the two △ s are equal in all respects.

(*Th. 18, Bk. I.*)

Hence, the whole quadl. ABCF *minus* the △FDC

= the whole quadl. ABCF *minus* the △EAB.

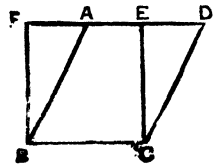
(*Ax. 3.*)

∴ the par^m. ABCD = the par^m. EBCF.

Q. E. D.

Cor. *The area of a parallelogram is equal to that of a rectangle contained by the base and altitude of the parallelogram.*

Let ABCD be a par^m. on the base BC. Draw CE ⊥ to AD; then EC is the altitude of the par^m.

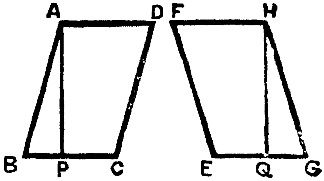


Draw BF ∥ to CE, meeting DA produced in F. Then FBCE is a rectangle, and it is contained by BC and CE.

Now, since FC and AC are par^{ms}. on the same base and between the same parallels, ∴ they are equal in area; which proves the proposition.

Note. *Hence parallelograms on equal bases and of equal altitudes are equal in area.*

Let ABCD, FEGH be two par^{ms}. on equal bases BC and EG, and having equal altitudes AP and HQ.



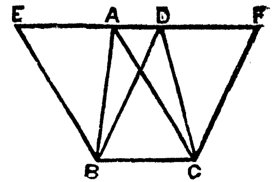
Now, the par^m. ABCD = the rect. BC, AP;
and the par^m. FEGH = the rect. EF, HQ.

But BC.AP = EG.HQ;

∴ the par^m. ABCD = the par^m. FEGH.

Theorem 2. (Euc. I. 37.)

Triangles on the same base and between the same parallels are equal in area.



Let the $\triangle$s ABC, DBC be on the same base BC and between the same $\parallel$s AD, BC.

To prove that the $\triangle$ s ABC, DBC are equal in area.

Draw BE $\parallel$ to CA, meeting DA produced in E; and draw CF $\parallel$ to BD, meeting AD produced in F.

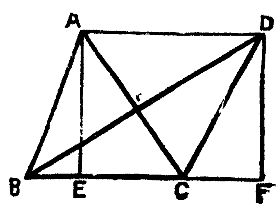
Proof. EBCA and DBCF are par$^{\text{ms}}$.; and they are on the same base BC, and between the same $\parallel$s EF, BC.

$\therefore$ the par$^{\text{m}}$. EBCA = the par$^{\text{m}}$. DBCF.

But, the $\triangle$ ABC = half the par$^{\text{m}}$. EBCA; and the $\triangle$ DBC = half the par$^{\text{m}}$. DBCF. } (*Th. 25. Bk. I.*)

$\therefore$ the $\triangle$ ABC = the $\triangle$ DBC. (*Ax. 7.*)

Q. E. D.



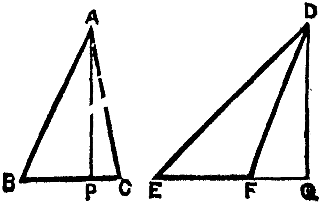
Cor. *Triangles on the same base and of the same altitude are equal in area.*

Let ABC and DBC be two $\triangle$s on the same base BC and having equal altitudes AE, DF.

Now AE is $\parallel$ to DF, and also equal to it: $\therefore$ AD is $\parallel$ to EF, *i.e.*, to BC. (*Th. 22, Bk. I.*)

Hence the $\triangle$s ABC and DBC are in the same base BC and *between the same* $\parallel$s;

$\therefore$ the $\triangle$ ABC = the $\triangle$ DBC.



Note 1. *Triangles on equal bases and of the same altitude are equal in area.*

Let ABC and DEF be two Δs on equal bases BC and EF, and having equal altitudes AP, DQ.

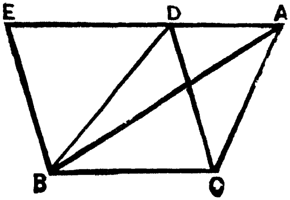
Suppose the Δ **DEF** taken up and so placed that **EF** may coincide with **BC**, and **D** may fall on the same side of **BC** as **A**. Then the two Δs will be on the same base and of the same altitude, and hence they are equal in area.

Note 2. Hence triangles on equal bases and *between the same parallels* are equal in area. For, triangles between the same parallels are of the same altitude.

Note 3. *Any median of a triangle divides the triangle into two equal parts.* For, the triangles on the two sides of the median are on equal bases and have a common altitude.

Theorem 3. (Euc. I. 41.)

If a triangle and a parallelogram be on the same base and between the same parallels, then the area of the angle is equal to half that of the parallelogram.



Let the △ABC and the par^{m}. BCDE be on the same base BC, and between the same ∥s EA, BC.

To prove that the △ABC is half of the pal^{m}. BCDE.

Proof. Join BD.

The digonal BD bisects the par^{m}. EC;

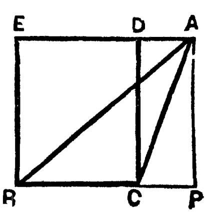
∴ the △DBC is half of the par^{m}. BCDE.

But the △DBC = the △ABC; because they are on the same base BC, and between the same ∥s BC, DA;

∴ the △ABC is half of the par^{m}. BCDE.

Q. E. D.

Cor. *The area of a triangle is equal to half that of a rectangle contained by the base and altitude of the triangle.*



Let ABC be a △ of which BC is the base and AP, the altitude.

Draw AE ∥ to PB; also draw CD and BE each ∥ to PA, meeting AE in D and E respectively.

Then the par^m. EC and the $\triangle$ ABC are on the same base and between the same $\|$s, $\therefore$ the $\triangle$ABC $= \frac{1}{2}$ of the par^m. EC.

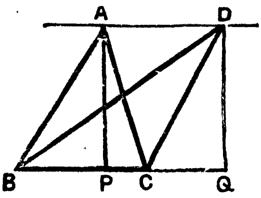
But, since CD is $\|$ to PA, the par^m. EC is evidently a *rectangle*; and it is contained by BC and CD, *i.e.*, by BC and PA.

Hence the $\triangle$ ABC $= \frac{1}{2}$ the rect. contained by BC and AP.

Note. Hence if the base and altitude of a triangle are respectively equal to b and h units of length, then the area of the $\triangle = \frac{1}{2}bh$ units of area.

Theorem 4.

Equal triangles on the same base are of the same altitude.



Let the $\triangle$s ABC and DBC, standing on the same base BC, be equal in area.

Let AP and DQ be the altitudes of the $\triangle$s ABC and DBC.

To prove that AP = DQ.

Proof. The $\triangle$ABC = $\frac{1}{2}$ the rect. BC, AP ; } (*Th. 3, Cor.*)
and the $\triangle$DBC = $\frac{1}{2}$ the rect. BC, DQ. }

$\therefore$ the rect. BC, AP = the rect. BC, DQ.

$\therefore$ AP = DQ. Q. E. D.

Note. It may be similarly proved that *equal triangles on equal bases are of the same altitude.*

Cor. *Equal triangles on the same base (or on equal bases in the same straight line) and on the same side of it are between the same parallels.*

For in the above diagram, AP being equal and ∥ to DQ, AD is ∥ to PQ, *i.e.*, to BC.

EXERCISE (13).

1. Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area.

2. Prove that equal triangles on opposite sides of the same base have the same altitude. Hence prove that if

two equal triangles ABC, DBC be on opposite sides of the same base BC, AD is bisected by BC or BC produced.

3. BC and AD are the parallel sides of a trapezium and E is the mid-point of CD. Prove that the $\triangle AEB$ is half of the whole figure.

4. If two triangles have two sides of the one equal to two sides of the other, and the contained angles supplementary; shew that the two triangles can be so placed at to form two parts of the same triangle, and hence tha they are equal in area.

5. If two sides of a triangle are given, prove that the area is greatest when the included angle is a right angle.

6. Prove that if a quadrilateral be bisected by each of its diagonals, it must be a parallelogram.

7. Prove that in any quadrilateral the straight lines joining the middle points of opposite sides bisect each other.

8. If E, F be the mid-points of the sides AC, AB of a triangle ABC, prove that the triangle AEF is one-fourth of the whole triangle ABC.

9. D, E are the mid-points of the sides AB, AC of a triangle ABC; and P is any point in BC produced. Prove that the triangle PDE is one-fourth of the triangle ABC.

10. Find the area of a triangle whose base and altitude are 3·5 and 2·8 inches respectively.

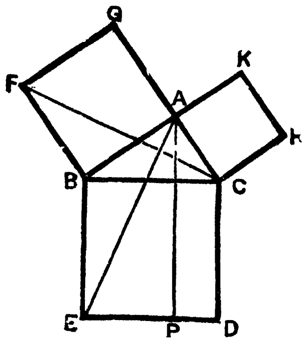
11. Find the altitude of a triangle whose area is 3·75 sq. inches and the base, 3 inches.

12. Find the base of a triangle whose area is 19·25 sq, inches and the altitude, 7 inches.

13. If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and the included angles supplementary, prove that the parallelograms are equal in area.

Theorem 5. (Euc. I. 47.)

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.



Let ABC be a right-angled △, having the right angle at A. Let BCDE, ABFG and ACHK be the squres described on BC, AB and AC respectively.

To prove that the sq. BD = the sum of the squares AF and AH.

Draw AP ∥ to BE or CD, meeting DE in P.

Join AE, CF.

Proof. Since each of the adj. ∠s BAC, BAG is a rt. ∠;

∴ CA and AG are in one str. line.

Now, the ∠ CBE = the ∠ ABF ; (*each being a rt.* ∠)

∴ adding the ∠ ABC to each of them, we have the whole ∠ ABE = the whole ∠ FBC.

Then in the two △s ABE, FBC, we have

(1) AB = FB,

(2) BE = BC,

and (3) the ∠ ABE = the ∠ FBC :

∴ the two △s are equal in all respects. (*Th. 4, Bk. I.*)

Now, the $\triangle$ABE and the rect. BP are on the same base BE and between the same $\|$s AP, BE ;

$\therefore$ the rect. BP = twice the $\triangle$ABE. (*Th.* 3.)

Also, since the $\triangle$FBC and the sq. AF are on the same base FB and between the same $\|$s GC, FB,

$\therefore$ the sq. AF = twice the $\triangle$ FBC. (*Th.* 3.)

Hence the rect. BP = the sq. AF. (*Ax.* 6.)

Similarly, by joining AD and BH, it may be shewn that the rect. CP = the sq. AH.

$\therefore$ the rectangles BP and CP are together

= the sum of the squares AF and AH ;

i.e., the sq. BD = the sum of the sqs. AF and AH.

Q. E. D.

Note 1. The above result may be briefly written as follows ; $BC^2 = CA^2 + AB^2$.

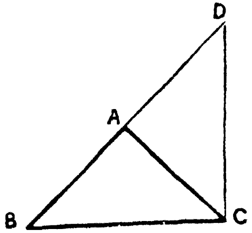
Note 2. If the sides BC, CA, AB are respectively equal to *a*, *b*, *c*, units of length, then

$$a^2 = b^2 + c^2.$$

$$\left.\begin{array}{ll} \text{Hence} & b^2 = a^2 - c^2. \\ \text{and} & c^2 = a^2 - b^2. \end{array}\right\}$$

Theorem 6. (Euc. I. 48.)

If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, then the angle contained by those two sides is a right angle.



Let ABC be a triangle such that $BC^2 = CA^2 + AB^2$.

To prove that the $\angle$ BAC is a right angle.

Draw AD $\perp$ to AC, and make AD = AB.

Join DC.

Proof. Since the $\angle$ CAD is a rt. $\angle$,

$$\therefore \quad DC^2 = CA^2 + AD^2 \qquad (Th.\ 5.)$$
$$= CA^2 + AB^2.$$

Hence the sq. on DC = the sq. on BC;

$$\therefore \quad DC = BC.$$

Now, in the two $\triangle$s ABC, ADC, we have

(1) AB = AD, *(Cons.)*

(2) AC common,

and (3) BC = DC,

$\therefore$ the two $\triangle$s are congruent. *(Th. 7, Bk. I.)*

Hence the $\angle$ BAC = the $\angle$ DAC = a rt. $\angle$. Q. E. D.

Note 1. We have $3^2 + 4^2 = 5^2$. Hence if the sides of a triangle are 3, 4 and 5 units of length respectively, the triangle is right-angled.

Note 2. Again, since $5^2 + 12^2 = 13^2$, $\therefore$ the triangle whose sides are 5, 12 and 13 units of length respectively, is right-angled.

Note 3. We have $(m^2-n^2)^2+(2mn)^2=(m^2+n^2)^2$. Hence, generally speaking, if the sides of a triangle are (m^2-n^2), $2mn$ and (m^2+n^2) units of length respectively, the triangle is right-angled.

Hence for each pair of numbers that may be substituted for m and n, we shall get a set of three numbers representing three str. lines which can form a right-angled triangle. Thus if $m=2$ and $n=1$, we have the sides equal to 3, 4 and 5 units respectively; if $m=4$ and $n=3$, we have the sides equal to 7, 24 and 25 units respectively; if $m=4$ and $n=1$, we have the sides equal to 15, 8 and 17 units respectively; and so on.

EXERCISE (14).

1. Construct a square whose area is double that of a given square.

2. Construct a square whose area is treble that of a given square. Hence shew how to construct a square whose area is seven times that of a given square.

3. A and B are two fixed points; and CD is any straight line perpendicular to AB, meeting AB or AD produced in D. If P be *any* point in CD, prove that the difference of the squares on PA and PB is equal to the difference of the squares on DA and DB.

4. Prove that a triangle whose sides are respectively 28, 45 and 53 units of length, is a right-angled triangle.

5. If a man starting from a place A travels 12 miles due North and then 5 miles due East, how far is he from A at the end of his journey?

6. A ladder 37 feet long rests with one end on the ground and the other on the top of a wall. Calculate the height of the wall if the distance of the foot of the ladder from the wall be 12 feet.

7. A rod 13 feet long is held vertically in a tank of water, one end of the rod reaching the bottom of the tank, and the other being *above* the surface of the water. Another rod of equal length is held in a slant position, one end being in contruct with the lower end of the first rod and the other being *on the surface* of the water. If the upper end of the second rod be 5 feet distant from the point where the first cuts the surface of water, calculate the length of that portion of the first rod which is above the water.

8. Divide a given straight line into two parts so that the sum of the squares on the parts may be equal to a given square.

9. Divide a given straight line into two parts so that the difference of the squres on the parts may be equal to a given square.

10. Find a straight line the square on which is equal to the difference of two given squares.

11. AD is the altitude of an equitateral triangle ABC. If AD and BD respectively contain p and m units of length, prove that $p^2 = 3m^2$.

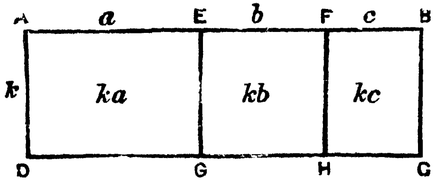
12. ABC is a triangle acute-angled at B. Prove that AC^2 is less than $AB^2 + BC^2$.

Hence prove that if in a triangle ABC, $AC^2 > AB^2 + BC^2$, then the angle ABC is obtuse.

Theorem 7. (Euc. II. 1.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$k(a+b+c)=ka+kb+kc.$$



Let ABCD be a rectangle of which the side AB is divided into any number of parts AE, EF and FB.

Suppose that AD, AE, EF and FB are respectively equal to k, a, b and c units of length.

Draw EG and FH ∥ to AD, meeting DC in G and H.

Then the figures AG, EH and FC are rectangles and their areas are respectively equal to ka, kb and kc units of area.

And the area of the whole rectangle AC

$= k(a+b+c)$ units of area.

But the whole rectangle = the sum of its parts ;

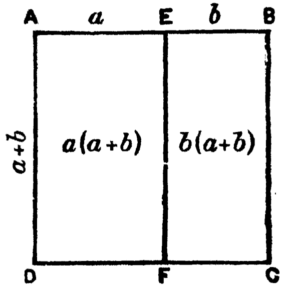
$$\therefore \quad k(a+b+c) = ka+kb+kc.$$

Thus, *if there are two straight lines of which one is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.*

Theorem 8. (Euc. II. 2.)

To illustrate and explain the geometrical theorem corresponding to the identity.

$$(a+b)^2 = a(a+b)+b(a+b).$$



Let ABCD be a square of which the side AB is divided into any two parts AE and EB.

Suppose that AE and EB are respectively equal to a and b units of length; then the side AD = $(a+b)$ units of length.

Draw EF ∥ to AD, meeting DC in F.

Then each of the figures AF and EC is a rectangle; and their areas are respectively equal to $a(a+b)$ and $b(a+b)$ units of area.

And the area of the whole figure AC

$$= (a+b)^2 \text{ units of area.}$$

But the whole figure is equal to the sum of its parts;

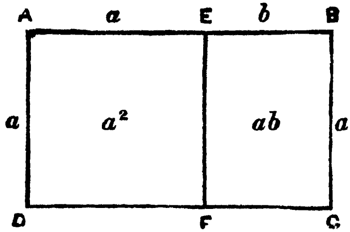
$$\therefore \quad (a+b)^2 = a(a+b)+b(a+b).$$

Thus, *if a straight line be divided into any two parts, the square on the straight line is equal to the sum of the rectangles contained by the whole line and each of the parts*

Th orem 9. (Euc. II. 3.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$a(a+b)=a^2+ab.$$



Let ABCD be a rectangle, of which the side AB is divided into two parts AE and EB, such that AE = AD.

Suppose that AE and EB are respectively equal to a and b units of length.

Draw EF ∥ to AD, meeting DC in F.

Then the figure AF is a square, and the figure EC is a rectangle ; their areas being respectively equal to a^2 and ab units of area.

And the area of the whole figure AC

$$=a(a+b) \text{ units of area.}$$

But the whole figure is equal to the sum of its parts ;

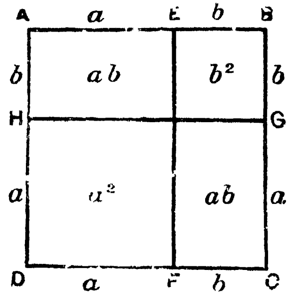
$$\therefore \quad a(a+b)=a^2+ab.$$

Thus, *if a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.*

Theorem 10. (Euc. II. 4.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$(a+b)^2 = a^2 + 2ab + b^2.$$



Let ABCD be a square of which the side AB is divided into any two parts AE and EB ; and suppose that these two parts are respectively equal to a and b units of length.

Draw EF ∥ to AD, meeting DC in F. From BC cut off BG = BE, and draw GH ∥ to BA meeting AD in H.

Then the figures HE and FG are rectangles, and the area of each of them = ab units of area ; and the figures EG, FH are squares, their areas being respectively equal to b^2 and a^2 units of area.

Also the area of the whole figure AC
$= (a+b)^2$ units of area.

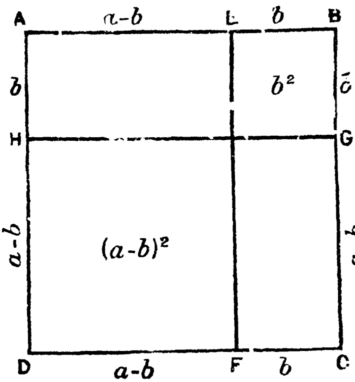
But the whole figure is equal to the sum of its parts ;
$\therefore\ (a+b)^2 = a^2 + b^2 + 2ab.$

Thus, *if a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the parts.*

Theorem 11. (Euc. II. 7.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$(a-b)^2 = a^2 - 2ab + b^2.$$



Let ABCD be a square of which the side AB is divided into two parts AE and EB.

Suppose that the whole str. line AB and its part BE are respectively equal to a and b units of length; then the part AE $=(a-b)$ units of length.

Draw EF ‖ to AD, meeting DC in F; cut off BG = BE, and draw GH ‖ to BA, meeting AD in H,

Then the areas of the rectangles AG, EC are each $= ab$ units of area; and the areas of the squares HF and EG are respectively $=(a-b)^2$ and b^2 units of area.

Also the area of the whole square AC $= a^2$ units of area.

Now, the sq. HF + the rect. AG + the rect. FG
= the sq. AC.

Adding the sq. EG to each of these equals we have the sq. HF + the rect. AG + the rect. EC
= the sq. AC + the sq. EG ;

i. e, $(a-b)^2+2ab=a^2+b^2$;

or, $(a-b)^2=a^2+b^2-2ab$.

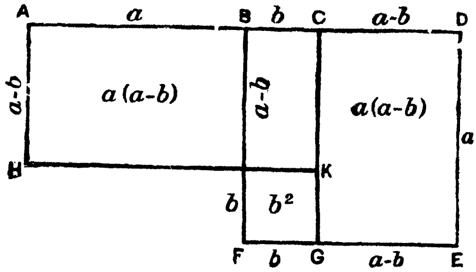
Thus, *the square on the difference of any two straight lines is equal to the sum of the squares on the lines, diminished by twice the rectangle contained by them.*

Note. In the above diagram, the str. line AE may be said to be divided *externally* at B into the segments AB, BE. Hence the above theorem may also be enunciated as follows: *If a straight line be divided externally at any point, the square on the line is equal to the sum of the squares on the two segments diminshed by twice the rectangle contained by them.*

Theorem 12. (Euc. II. 5 and 6.)

To illustrate and explain the geometrical theorem corresponding to the identity.

$$a^2 - b^2 = (a+b)(a-b).$$



Let the two str. lines AB and BC be in the same str. line, and let them contain a and b units of length respectively.

Produce BC to D making BD = AB; then CD = $(a-b)$ units of length.

Let BDEF be the square on BD;

draw CG ‖ to BF, meeting FE in G;

draw AH ‖ to BF, making AH = CD;

and draw HK ‖ to AC, meeting CG in K.

The rectangles HB and CE are equal; the area of the rect. AK = $(a+b)(a-b)$ units of area; and the areas of the squares BE and FK are respectively = a^2 and b^2 units of area.

Now, the difference of the two squares

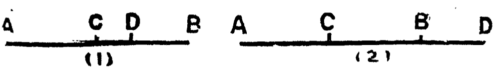
= the rect. CE + the rect. BK

$$= \text{the rect. } \mathsf{HB} + \text{the rect. } \mathsf{BK}$$
$$= \text{the rect. } \mathsf{AK}.$$

Hence $a^2 - b^2 = (a+b)(a-b)$.

Thus, *the difference of the squares on any two striaght lines is equal to the rectangle contained by their sum and difference.*

Note. *If a straight line be bisected, and also divided into two unequal segments (either internally or externally), then the rectangle contained by the unequal segments is equal to the difference of the squares on half the line and on the line between the points of section.*



For, if the str. line **AB** be bisected at **C** and also divided into two unequal segments at **D**, then in fig. (1) we have

$$\mathsf{AD.\,DB} = (\mathsf{AC}+\mathsf{CD})(\mathsf{CB}-\mathsf{CD})$$
$$= (\mathsf{AC}+\mathsf{CD})(\mathsf{AC}-\mathsf{CD})$$
$$= \mathsf{AC}^2 - \mathsf{CD}^2;$$

and in fig. (2)

$$\mathsf{AD.\,DB} = (\mathsf{CD}+\mathsf{AC})(\mathsf{CD}-\mathsf{CB})$$
$$= (\mathsf{CD}+\mathsf{AC})(\mathsf{CD}-\mathsf{AC})$$
$$= \mathsf{CD}^2 - \mathsf{AC}^2.$$

EXERCISE (15).

1. Illustrate and explain the geometrical theorem corresponding to the identity

$$(2a)^2 = 4a^2.$$

2. Illustrate and explain the geometrical theorem corresponding to the identity

$$(3a)^2 = 9a^2.$$

3. Prove geometrically that

$$(a+b)(c+d) = ac + ad + bc + bd.$$

4. Prove geomatrically that

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca.$$

5. Prove geometrieally that

$$(a-b)(a-c) = a^2-ac+bc-ab.$$

6. Illustrate and explain the geometrical theorem corresponding to the identity

$$(a+b)(b+c) = ac+b(a+b+c).$$

7. If AD is the perpendicular upon the hypotenuse BC of a right-angled triangle ABC, prove that

$$AD^2 = BD.\ DC.$$

Hence prove that (i) $AB^2 = BC.\ BD$, and (ii) $AC^2 = BC.\ CD$.

8. ABC is a triangle. and AD is perpendicular to BC. If $AD^2 = BD.\ DC$, prove that BAC is a right angle.

9. C is any point in a given straight line AB. Deduce from Theorems 9 and 10, that

$$AB^2+BC^2 = AC^2+2AB.\ BC.$$

10. If C is the mid-point of a given straight line AB, and if D is any point in CB, prove that

$$AD^2+DB^2 = 2AC^2+2CD^2.$$

11. If C is the mid-point of a given straight tine AB, and if D is any point in CB produced, prove that

$$AD^2+DB^2 = 2AC^2+2CD^2.$$

12. C is the mid-point of a given straight iine AB. If a point D moves from B to A, prove that the area of the rect. AD. DB gradually *increases* until D coincides with C, when it has its *greatest* value, and thereafter gradually *diminishes*.

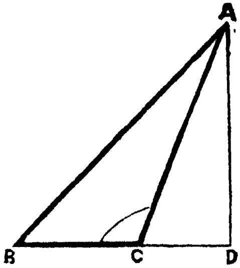
13. In the previous case, prove that $AD^2 + DB^2$ gradually *diminishes* until B coincides with C, when it has its *least* value, and thereafter gradually *increases*.

Hence shew that if the sum of two straight lines be given, the least value of the sum of the squares on the lines is equal to half the sqare on the given sum.

14. AB is a given straight line. If a point C moves from B to A, prove that $AB^2 + BC^2$ is always *greater than* 2AB.BC, except when C coincides with A; and that in this case $AB^2 + BC^2 = 2AB.BC$.

Theorem 13. (Euc. II. 12.)

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the other two sides together with twice the rectangle contained by one of these two sides and the projection on it of the other.



Let ABC be a $\triangle$ obtuse-angled at C; and let AD be $\perp$ to BC produced, so that CD is the projection of AC on BC.

To prove that

$$AB^2 = BC^2 + CA^2 + 2BC.CD.$$

Proof. $BD^2 = BC^2 + CD^2 + 2BC.CD.$ (*Th. 10.*)

To each of these equals add DA^2.

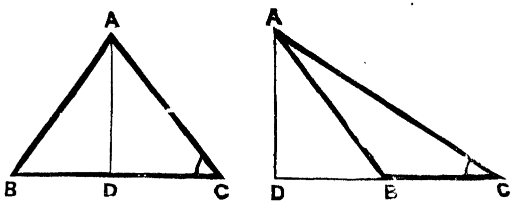
Then $BD^2 + DA^2 = BC^2 + CD^2 + DA^2 + 2BC.CD.$

But $\left.\begin{array}{l} BD^2 + DA^2 = AB^2, \\ \text{and } CD^2 + DA^2 = AC^2. \end{array}\right\}$ (*Th. 5.*)

$\therefore AB^2 = BC^2 + CA^2 + 2BC.CD.$ Q. E. D.

Theorem. 14. (Euc. II. 13.)

In any triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by one of these two sides and the projection on it of the other.



Let ABC be any △ acute-angled at C; and let AD be ⊥ to CB or CB produced, so that CD is the projection of AC on BC.

To prove that

$$AB^2 = BC^2 + CA^2 - 2BC.CD.$$

Proof. Since BD is the difference of the two str. lines BC, CD, we have

$$BD^2 = BC^2 + CD^2 - 2BC\ CD. \qquad (\textit{Th. 11.})$$

To each of these equals add DA^2.

Then $BD^2 + DA^2 = BC^2 + CD^2 + DA^2 - 2BC.CD.$

But $\left.\begin{aligned} BD^2 + DA^2 &= AB^2, \\ \text{and}\quad CD^2 + DA^2 &= AC^2. \end{aligned}\right\}$ (*Th. 5.*)

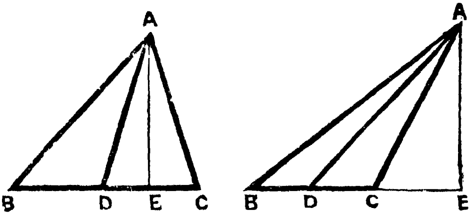
$\therefore\quad AB^2 = BC^2 + CA^2 - 2BC.CD.$ Q. E. D.

Note. Comparing Theorems 5, 13 and 14, we come to the following conclusion :—

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse, right, or acute; the difference in the cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

Theorem 15.

The sum of the squares on any two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.



Let AD be the median which bisects the side BC of a $\triangle$ ABC.

To prove that $AB^2 + AC^2 = 2BD^2 + 2DA^2$.

Proof. Let AE be $\perp$ to BC or BC produced; then in the above diagram, the $\angle$ ADB is obtuse and $\therefore$ the $\angle$ ADC is acute.

Hence in the $\triangle$ ABD,

$$AB^2 = BD^2 + DA^2 + 2BD \cdot DE; \qquad \text{(Th. 13.)}$$

and in the $\triangle$ ADC,

$$AC^2 = CD^2 + DA^2 - 2CD \cdot DE \qquad \text{(Th. 14.)}$$
$$= BD^2 + DA^2 - 2BD \cdot DE.$$

Hence by addition.

$$AB^2 + AC^2 = 2BD^2 + 2DA^2. \qquad \text{Q. E. D.}$$

EXERCISE (16).

1. The numerical measures of the sides of a triangle are 7, 9 and 11 : prove that the angle opposite to the greatest side is neither right nor obtuse.

Construct the triangle, taking one-tength of an inch as the unit of length.

2. The numerical measures of the sides of a triangle are 8, 13 and 17 : shew that the angle, opposite to the greatest side must be obtuse.

Construct the triangle, taking three-tenths of a centimetre as the unit of length.

3. ABC is an acute-angled triangle. The perpendiculars from A, B, C upon the opposite sides meet at O. Prove that

$$OA^2 + OB + OC^2 < \tfrac{1}{2}(AB^2 + BC^2 + CA^2).$$

4. The numerical measure of a side of an equilateral triangle ABC is a. BC is produced to D so that CD = 2BC. If x be the numerical measure of AD, prove that $x^2 = 7a^2$.

5. ABC is an acute-angled triangle; and BE, CF are perpendiculars upon CA, AB. Prove that AB. AF = AC. AE.

6. D is the mid-point of the side BC of a triangle ABC; and AE is perpendicular to BC, or BC produced. Use Theorems 13 and 14 to prove that the difference of the squares on AB, AC is equal to twice the rect. BC.DE.

7. C is the mid-point of a given straight line AB. If a point P moves on the circumference of a circle of which the centre is C, prove that $PA^2 + PB^2$ is always the same.

8. The distance between two fixed points A and B is 6 inches. If a point P moves so that $PA^2 + PB^2$ is always $= 50$ square inches, find the locus of P.

9. Prove that the sum of the squares on the two equal sides of an isosceles triangle is less than the sum of the squares on the two sides of any other triangle on the same base and between the same parallels.

10. ABCD is a rectangle; and P is any point, either inside or outside it. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

11. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

12. Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

13. If O is the centroid of a $\triangle ABC$, prove that

$$3(OA^2 + OB^2 + OC^2) = AB^2 + BC^2 + CA^2.$$

14. Prove that the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals together with four times the square on the line joining the middle points of the diagonals.

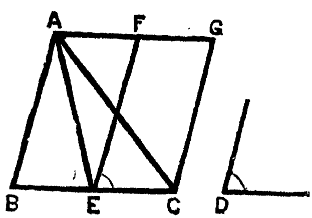
Hence prove that if the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral must be a parallelogram.

SECTION III.

PROBLEMS.

Problem 1. (EUC. I. 42.)

To construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.



Let ABC be a given $\triangle$, and D the given $\angle$.

It is required to construct a par^m. equal to the $\triangle$ABC, and having one of its $\angle$s = the $\angle$D.

Cons. Bisect BC at E.

At E in CE, make the $\angle$CEF = $\angle$D ; draw CG $\parallel$ to EF ; draw AG $\parallel$ to BC, meeting EF and CG in F and G respectively.

Then the figure FC is the required par^m.

Proof. Join AE.

Since the two $\triangle$s ABE, AEC are on equal bases BE, EC, and of the same altitude ;

∴ the $\triangle$ABE = the $\triangle$AEC.

∴ the $\triangle$ABC is double of the $\triangle$AEC. ... ... (α)

The quadl. FC is a par^{m}., because its opposite sides are $\parallel$.

Now, the par^{m}. FC and the $\triangle$AEC are on the same base EC and between the same $\parallel$s AG, EC ;

∴ the par^{m}. FC is double of the $\triangle$AEC. ... ...(β)

Hence, from (α) and (β),

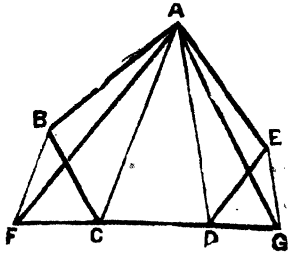
the par^{m}. FC = the $\triangle$ABC ;

and it has also the $\angle$ CEF = the $\angle$ D. Q. E. F.

Note. If the given $\angle$ D be a right angle, the parallelogram constructed will be a *rectangle*.

Problem 2.

To construct a triangle equal in area to any given rectilineal figure.



Let ABCDE be a given rectilineal figure.

's required to construct a △ equal in the rectil. figure ABCDE.

'n AC, AD.

AC, meeting DC produced in F ; and 'ting CD produced in G.

AG.

iired △.

BC, AFC are on the same base ||s AC, BF ;

AFC. ... ... (α)

n the same base AD

... (β)

Hence, from (α) and (β),

$$\text{the } \triangle AFC + \text{the } \triangle ACD + \text{the } \triangle AGD$$
$$= \text{the } \triangle ABC + \text{the } \triangle ACD + \text{the } \triangle AED\ ;$$

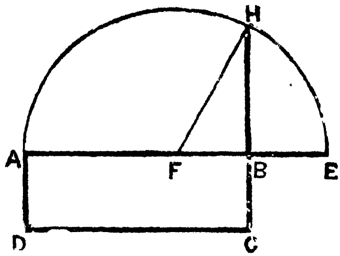
i.e., the $\triangle AFG$ = the rectil. figure ABCDE. Q. E. F.

Note 1. The given rectil. figure is = the rectil. figure ABCG which has one side less than those of the given figure. From this it is clear that a *seven* sided rectil. figure can be reduced to one of *six* sides, and this latter again to one of *five* sides ; and so on. Thus, by successive repetitions of the above process, any given rectil. figure may be finally reduced to a triangle of equal area.

Note 2. We can construct a *rectangle* equal to any given rectil. figure. For, all that we have to do is to reduce the given figure to a triangle of equal area, and then construct a rectangle equal to this triangle.

Problem 3.

To construct a square equal in area to a given rectangle.



Let ABCD be the given rectangle, of which the side
ꟷB is > the side BC.

ᵀt is required to construct a square
ˋ to the rectangle AC.

Produce AB to E, making BE = BC.

mid-pt. of AE ; with centre F, and radius
a semi-circle on AE.

ʹ the O^{ce} in H.
is the required square.

ʳt. ∠.

(*Th.* 5.)

(*Th.* 12.)

ꟷ. F.

Note. We can now construct a square equal to any given rectil. figure. For, all that we have to do is to construct a *rectangle* equal to the given figure (Note 2, Prob. 2), and then construct a square equal to this rectangle.

EXERCISE (17).

1. On one side of a triangle construct a parallelogram equal to the triangle.

2. On one side of a triangle as base construct an isosceles triangle, equal to half the given triangle.

Hance shew how to construct a *rhombus* equal to a given triangle, having one side of the triangle as a diogonal.

3. Bisect a parallelogram by a straight line drawn through any given point.

4. Bisect a triangle by a straight line drawn through a given point in one of its sides.

5. Bisect a quadilateral by a straight line drawn through one of its angular points.

6. P is a given point in the side AB of a triangle ABC. Find a point D in BC produced so that the triangle PBD may be equal to the triangle ABC.

7. ABC is a triangle and P is a point in BA produced. Find a point D in BC so that the triangle PBD may be equal to the triangle ABC.

8. Divide a triangle into fiv equal parts by straight lines drawn through one of ... nts.

Hence cut off f ... of it by a straight l ... its sides.

9. Trisect a triangle by straight lines drawn through a given point in one of its sides.

10. On a given straight line AB construct a rectangle equal to a given square.

11. On a given straight line AB construct a rectangle equal to a given rectangle.

12. On a given straight line construct a rectangle equal to a given triangle.

13. On a given straight line construct a triangle equal to a given triangle.

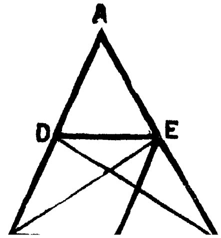
14. Shew that if a rectangle be equal to a square, the perimeter of the square is less than that of the rectangle.

Hence shew that of all rectangles having the same area the square is that which has the *least* perimeter.

SECTION IV.

MISCELLANEOUS PROPOSITIONS.

1. *The straight line joining the middle points of any two sides of a triangle is parallel to the third side, and equal to half of it.*



Let D, E be the mid-pts. of the sides AB and AC of a △ ABC.

To prove that DE is ∥ to BC and is $= \frac{1}{2}$BC.

Proof. (1) Join BE, CD.

The △s BDC, ADC are on equal bases and of the same altitude ;

∴ the △ BDC = the △ ADC ;

and ∴ the △ BDC $= \frac{1}{2}$ the △ ABC.
Similarly, the △ BEC $= \frac{1}{2}$ the △ ABC.

Hence the △ BDC = the △ BEC.

∴ DE is ∥ to BC. (*Cor. Th. 4.*)

(2) Let F be the mid-pt. of BC; join EF.

Then since E and F are the mid-pts. of AC and BC,

∴ EF is ∥ to AB.

Hence DF is a par^{m}.;

∴ DE = BF $= \frac{1}{2}$ BC. Q. E. D.

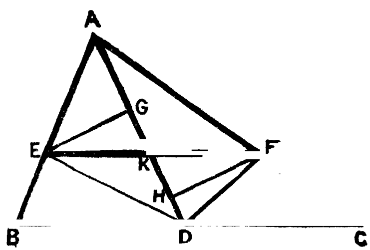
Cor. *Conversely, if through the middle point of a side of a triangle a straight line be drawn parallel to another, it will pass through the middle point of the third side.*

In the above diagram, if DE be ‖ to BC, then the △ BEC = the △ BDC.

∴ the △ BEC = ½ the △ ABC.

Hence the pt. E is the mid-pt. of AC (for, if any other pt. E′ were the mid-pt., the △ BE′C would be half of the △ ABC, which is impossible.

2. *If* D *is the mid-pt. of the side* BC *of a triangle* ABC, *prove that* AD *bisects any straight line* EF *drawn parallel to* BC *meeting the sides* AB, AC *at* E *and* F *respectively.*



Proof. (i) Join DE, DF.

The △s BED, DFC are on equal bases BD, DC and between the same ‖s EF, BC;

∴ the △ BED = the DFC. (*Note 2, Th. 2.*)

Also the △ ABD = the △ ADC.

∴ the △ AED = the △ AFD. (*Ax. 3.*)

(ii) Let EG and FH be the altitudes of the △s AED, AFD; and let K be the pt. where EF cuts AD.

The △ AED = ½ the rect. AD.EG;
and the △ AFD = ½ the rect. AD.FH. } (*Cor. Th. 3.*)

Hence $EG = FH$.

Now in the $\triangle$s EKG, FKH, we have

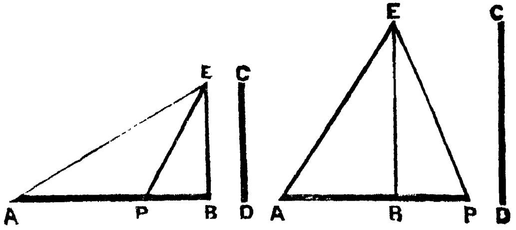
(1) the $\angle EKG =$ the $\angle FKH$,

(2) the $\angle EGK =$ the $\angle FHK$,

and (3) $EG = FH$.

Hence $EK = KF$. Q. E. D.

3. AB *is a given straight line. Find a point* P *in* AB, *or* AB *produced, such that the difference of the squares on* AP, BP *may be equal to the square on a given line* CD.



Cons. Draw BE $\perp$ to AB, making $BE = CD$.

Join AE ; and make the $\angle AEP = \angle EAB$.

Let EP meet AB, or AB produced, in P.

The point P thus found is the required point.

Proof. Since the $\angle AEP =$ the $\angle EAP$,

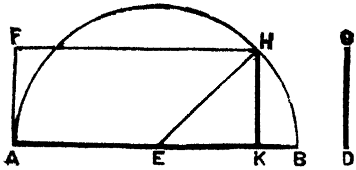
$\therefore \quad AP = EP$.

Hence, $AP^2 - BP^2 = EP^2 - BP^2$

$= EB^2$ ($\because$ the $\angle EBP$ is right)

$= CD^2$. Q. E. F.

4. *Divide a given straight line internally so that the rectangle contained by the two segments may be equal to a given square ; a side of the square being less than half the given stright line.*



Let AB be the given straight line, and CD a side of the given square.

To find a pt. K in AB **so that the rect.** AK.KB **may be = the sq. on** CD.

Cons. Bisect AB at E.

With E as centre and EA as radius, describe a semi-circle on AB.

Draw AF ⊥ to AB, making AF = CD.

Draw FH ∥ to AB, and let H be one of the points where FH cuts the semi-circle.

Draw HK ⊥ to AB ; then K is the reqd. pt.

Proof. Join EH.

FK is a rectangle ; ∴ HK = FA = CD.

Now, $AK.KB = AE^2 - EK^2$ (*Th. 12, Note.*)

$= EH^2 - EK^2$

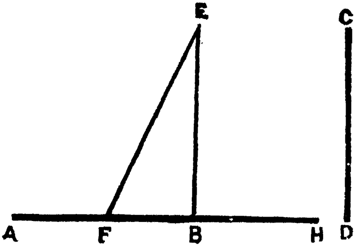
$= HK^2$

$= CD^2$. Q. E. F.

Note. Hence, given the area of a rectangle and the ***sum*** of its adjacent sides, we know how to construct the rectangle.

If the area be given as equal to that of a given ***rectangle***, we have first of all to construct a square equal to this rectangle (Problem 3), and then proceed as above.

5. *Divide a given straight line externally so that the rectangle contained by the two segments may be equal to a given square.*



Let AB be the given str. line, and CD a side of the given square.

To find a pt. H in AB produced so that the rect. AH. HB may be = the sq. on CD.

Cons. Draw BE ⊥ to AB, making BE = CD.

Bisect AB at F, and join FE.

Produce FB to H, making FH = FE.

Then H is the required pt.

Proof. AB is bisected at F and divided *externally* at H;

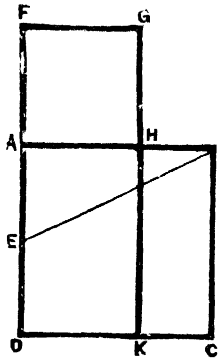
$$\therefore \quad AH.HB = FH^2 - FB^2 \qquad (\textit{Th. 12, Note.})$$
$$= FE^2 - FB^2$$
$$= EB^2$$
$$= CD^2.$$

Q. E. F.

Note. Hence, given the area of a rectangle and the *difference* of its adjacent sides, we know how to construct the rectangle.

If the area be given as equal to that of a given rectangle, we have first of all to construct a square equal to this rectangle (Problem 3), and then proceed as above.

6. AB *is a given straight line. Find a point* H *in* AB, *so that the rectangle contained by* AB *and* BH *may be equal to the square on* AH.



Cons. On AB describe the square ABCD.
Bisect AD at E, and join EB.
Produce EA to F, making EF = EB.
From AB cut off AH = AF; then H is the reqd. pt.

Proof. Through H draw GHK ∥ to FD, meeting DC in K; and draw FG ∥ to AB, meeting KG in G.

Now FK is a rectangle, which is contained by DF, FG and ∴ by DF, FA; and FH is the sq. on AH.

Since DA is bisected at E and divided *externally* at F,

$$\therefore \quad DF.FA = EF^2 - EA^2 \qquad (\textit{Th. 12, Note.})$$
$$= EB^2 - EA^2$$
$$= AB^2 \quad (\because \text{ the } \angle EAB \text{ is right}).$$

Thus, the rect. FK = the sq. AC.

Taking away the common part AK, we have

the sq. FH = the rect. HC

= the rect. AB, EH ($\because$ CB = AB).

That is, $AH^2 = AB.BH$. Q. E. F.

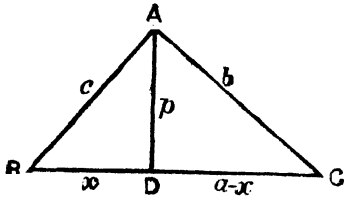
Note. A straight line is divided in **medial section** when the rectangle contained by the whole line and one of the parts is equal to the square on the other part.

7. *If the sides* BC, CA, AB *of a triangle be respectively equal to a, b, and c units of length, and if*

$$2s = a + b + c,$$

prove that the area of the triangle

$$= \sqrt{s(s-a)(s-b)(s-c)} \text{ units of area.}$$



Let the altitude AD of the triangle contain p units of length.

Let BD be $= x$ units of length; then CD $= (a-x)$ units of length.

From the $\triangle$ ADB, we have $p^2 = c^2 - x^2$;

and from the $\triangle$ ADC,

$$p^2 = b^2 - (a-x)^2.$$

$$\therefore \quad c^2 - x^2 = b^2 - (a-x)^2.$$

whence $\quad x = \dfrac{a^2 - b^2 + c^2}{2a}$.

Hence, $\quad p^2 = c^2 - x^2$

$$= c^2 - \left(\frac{a^2 - b^2 + c^2}{2a}\right)^2$$

$$= \left\{c + \frac{a^2 - b^2 + c^2}{2a}\right\}\left\{c - \frac{a^2 - b^2 + c^2}{2a}\right\}$$

$$= \frac{(c+a)^2-b^2}{2a} \cdot \frac{b^2-(c-a)^2}{2a}$$

$$= \frac{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}{4a^2}.$$

Now, since $2s = a+b+c$,

$$\therefore \left.\begin{aligned} c+a-b &= (a+b+c)-2b = 2(s-b), \\ b+c-a &= (a+b+c)-2a = 2(s-a), \\ a+b-c &= (a+b+c)-2c = 2(s-c). \end{aligned}\right\}$$

$$\therefore \quad p^2 = \frac{4s(s-a)(s-b)(s-c)}{a^2}$$

$$\therefore \quad p = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

Hence, the number of units of area in the area of the $\triangle ABC$

$$= \frac{ap}{2} = \sqrt{s(s-a)(s-b)(s-c)}.$$

EXERCISE (18).

1. BCDE is a square on the hypotenuse BC of a right-angled triangle ABC. Prove that

$$AB^2 + AD^2 = AC^2 + AE^2.$$

2. O is a point within a triangle ABC, such that the three angles OAB, OBC, OCA, are equal to one another. Find the position of O.

3. If two triangles that are equal in area stand on the same base and on *opposite* sides of it, prove that the straight line joining their vertices is bisected by the base or the base produced.

Hence, prove that the medians of a triangle are concurrent.

4. E, F, G, H are the middle points of the sides AB, BC, CD, DA respectively of any quadrilateral ABCD. Shew that the quadrilateral EFGH is a parallelogram, and that its area is half that of ABCD.

5. If the acute angles of a right-angled triangle are $60°$ and $30°$ respectively, prove that the equilateral triangle described on the hypotenuse is equal to the sum of the equilateral triangles described on the other two sides.

6. If the angle C of a triangle ABC be $60°$, prove that

$$AB^2 = BC^2 + CA^2 - BC.CA.$$

7. If the angle C of a triangle ABC be $120°$, prove that

$$AB^2 = BC^2 + CA^2 + BC.CA.$$

8. In a triangle ABC the angles B and C are acute. If BE, CF be perpendiculars to AC, AB respectively, prove that

$$BC^2 = AB.BF + AC.CE.$$

9. Divide a given straight line into two parts so that the square on one of the parts may be double the square on the other.

10. Prove that in any quadrilateral the sum of the squares on the diagonals is equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

11. BC is the hypotenuse of a right-angled triangle ABC, and AD is perpendicular to BC. If b, c, h are the numerical measures of CA, AB, AD respectively, prove that

$$\frac{1}{h^2} = \frac{1}{b^2} + \frac{1}{c^2}.$$

12. Bisect a quadrilateral by a straight line drawn through a given point in one of its sides.

13. Taking any straight line as the unit of length, find a straight line of which the numerical measure is $\sqrt{6}$.

14. Calculate the area of a triangle the numerical measures of whose sides are 10, 17 and 21.

Hence find the numerical measure of the perpendicular on the greatest side from the opposite angular point.

15. The nnmerical measures of the parallel sides of a trapezium are 20 and 34, and those of the other two sides are 13 and 15. Calculate the area of the trapezium.

16. Describe a *rhombus* equal to a given rectilineal figure.

17. Construct a rectangle equal to the sum or difference of two given rectangles.

18. A and B are two fixed points. P and Q are two other points such that

$$PA^2 - PB^2 = QA^2 - QB^2.$$

Prove that the perpendiculars from P and Q upon AB are in the same straight line.

Hence, if A and B are two fixed points and a point P moves so that $PA^2 - PB^2$ is always the same, prove that the locus of P is a straight line perpendicular to AB.

19. The base BC of a triangle ABC is divided internally at D, so that

$m.BD = n.CD$; where m and n are integers.

Prove that

$$m.\ AB^2 + n.\ AC^2 = m.BD^2 + n.DC^2 + (m+n).\ AD^2.$$

20. The base BC of a triangle ABC is divided *externally* at D, so that

$m.\ BD = n.CD$; where m and n are integers.

Prove that

$$m.\ AB^2 - n.AC^2 = m.BD^2 - n.DC^2 + (m-n).\ AD^2.$$

21. If O be the centroid of a triangle ABC and P any other point, prove that

$$PA^2 + FB^2 + PC^2 = OA^2 + OB^2 + OC^2 + 3PO^2.$$

What point is that the sum of the squares of whose distances from the vertices of a given triangle is the *least*?

BOOK III

The Circle.

SECTION I.

FUNDAMENTAL IDEAS AND DEFINITIONS.

1. A **circle** is a plane figure bounded by one line which is such that all straight lines drawn to it from a certain point within the figure are equal to one another.

2. The bounding line of a circle is called its **circumference**.

3. The point within a circle from which all straight lines drawn to the circumference are equal is called the **centre** of the circle.

4. Any straight line drawn from the centre of a circle to the circumference is called a **radius** of the circle.

Note. 1. All radii of the same circle are equal to one another.

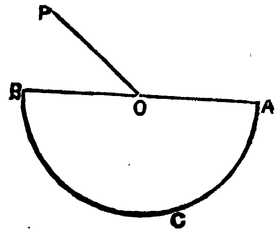
Note. 2. If a straight line be drawn from the centre of a circle equal to the radius, the extremity of the line will be on the circumference.

5. Any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a **diameter** of the circle.

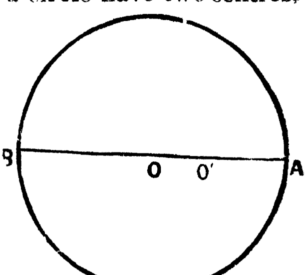
6. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference which it cuts off.

Note. 1. Every diameter divides a circle into two semi-circles.

Note. 2. Suppose the diameter AB of a semi-circle ACB to remain fixed, while the semi-circle is turned round AB until it falls on the side of AB remote from C. The distance of every point on the semi-circumference from O remains unaltered by this change of position. Hence, if OP be drawn in any direction on that side of AB which is remote from C and equal to a radius of the semi-circle, then P will be on the semi-circumference in its present position.



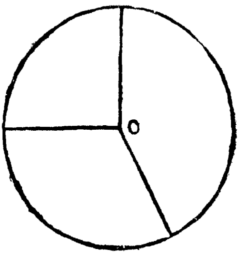
7. **A circle cannot have more than one centre.** If possible, let a circle have two centres, O and O′. Produce OO′ both ways to meet the O^{ce} in A and B. Then, since OA = OB, we have OA′ = $\frac{1}{2}$ AB; and similarly, O′A = $\frac{1}{2}$AB. Therefore OA = O′A, which is impossible. Hence, if O is the centre of a circle, there is *no other point* satisfying the same condition.



Note. *If the circumference of two circles coincide their centres must also coincide.* For, when the circumferences coincide, the two circles become one and the same circle. Hence, if the centres do not coincide, there will be two centres to one circle, which is impossible.

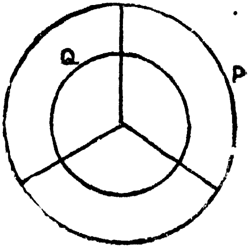
8. Two or more circles are said to be **concentric** when they have the same centre.

9. **If two circles have equal radii, they are also equal in area.** Suppose one of the circles to be taken up and so placed upon the other that the two circles become concentric, the point O being the common centre. Now, if a straight line be drawn from O in any direction and equal to the radius of one of the circles, it will also be equal to the radius of the other circle, and hence the extremity of the line will be on both the circumferences.



Thus, the two circumferences will coincide *every where*, and will consequently enclose the same amount of surface.

10. **If two circles have unequal radii, their areas also are unequal.** Suppose one of the circles to be taken up and so placed upon the other that the two circles become concentric. Let P denote the circle which has the larger radius, and Q the other circle.



Then, whichever point on the $\bigcirc^{ce}$ of Q may be taken, its distance from the common centre is less than the radius of P. Hence *every point on the* $\bigcirc^{ce}$ *of* Q, and consequently *the entire circumference, lies within the circle* P. Clearly therefore the amount of surface enclosed by the $\bigcirc^{ce}$ of Q is less than that enclosed by the $\odot^{ce}$ of P, which proves the proposition.

Note. Hence, *if two circles have equal areas their radii also are equal.* The raddi *cannot be unequal*, for, in that case, the areas would be unequal too.

11. The word "circle" is often used to mean the "circumference of a circle". For instance, when we say that "one circle cannot cut another in more than two points," we mean that "the circumference of one circle cannot cut the circumference of another in more than two points."

12. Any part of the circumference of a circle is called an **arc** of the circle.

13. The finite straight line joining any two points on the circumference of a circle is called a **chord** of the circle.

Note 1. A diameter is also a chord.

Note. 2. Every chord, which is not a diameter, divides the circumference into two *unequal* arcs, of which the greater one is called the **major arc**, and the lesser, the **minor arc.**

Note 3. Two arcs of a circle are said to be **conjugate** to each other when they together make up the whole circumference. Hence two *conjugate* arcs stand on opposite sides of the same chord.

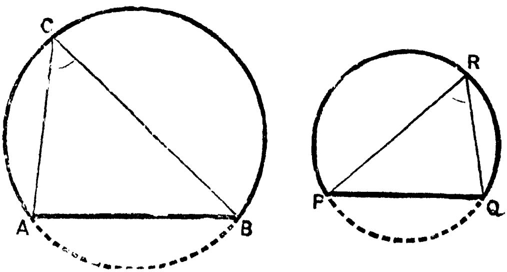
14. A **segment** of a circle is the figure bounded by an arc and the chord joining the extremities of the arc.

Note. 1. A semi-circle is also a segment.

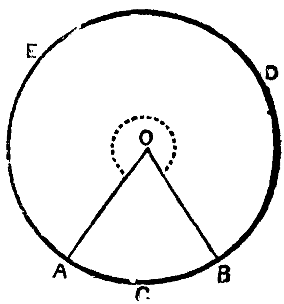
Note 2. The chord of a segment is also called the *base* of the segment.

15. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

16. **Similar segments** are those which contain equal angles. Thus, in the following diagram, the segments ACB and PRQ are *similar*, because the $\angle$ s ACB and PRQ are equal.



17. A **sector** of a circle is the figure bounded by an arc and the two radii drawn to the extremities of the arc.



Thus, in the adjoining diagram, the figure AOBC is a sector of the ⊙ABD. The figure AOBDE is also a sector.

Note 1. The angle between the two bounding radii of a sector is called the **angle of the sector.** Hence the angle of the sector AOBDE is the re-entrant angle AOB.

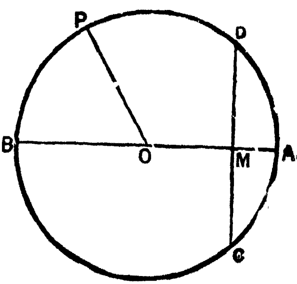
Note 2. A sector becomes a semi-circle when the angle of the sector becomes equal to two right angles.

18. A figure is said to be **symmetrical about a line** if the part of the figure on one side of the line coincides completely with the part on the other side when the figure is folded about that line.

Note 1. It is easy to see that an isosceles triangle is symmetrical about the bisector of its vertical angle.

Note 2. The line about which a figure is symmetrical is called **a line of symmetry** in the figure. In other words, if a straight line and a figure be so related that, when the figure is folded about the line, the part of the figure on one side of it coincides completely with the part on the other side, then the line is called ***a line of symmetry in the figure***.

19. A circle is symmetrical about any diameter. Let O be the centre of the circle ABC of which AB is a diameter, as in the adjoining diagram. Let the circle be folded about AB so that the semi-circumference ACB falls on the side of AB remote from C. Now, if from O a straight line OP be drawn in any direction to meet the semi-circumference ADB in P, then P will also be on the semi-circumference ACB *in its present position*, because OP is a radius of the circle.



Thus *any* straight line drawn through O to meet the semi-circumference ADB will meet both the semi-circumferences in the same point. Hence the two semi-circumferences will coincide *everywhere*.

Note 1. Two points on the circumference of a circle are said to be *symmetrically opposite* with regard to a diameter, if they coincide with one another when the circle is folded about that diameter. For instance, if the points C and D, in the above diagram, coincide when the circle is folded about AB, then C and D are *symmetrically opposite points* with regard to AB. Each of the points C and D is also said to be the *image* of the other with regard to the diameter BA.

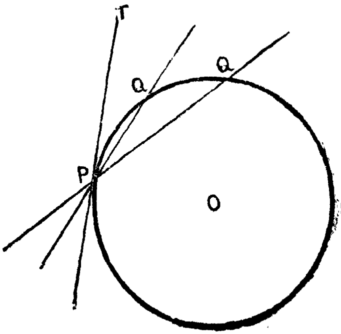
Note 2. *The chord joining two symmetrically opposite points with regard to a diameter is bisected at right angles by that diameter.*

In the preceding diagram, let **CD** cut **AB** at **M**. Then, when the circle is folded about **AB**, and **C** coincides with **D**, it is evident that **MC** coincides with **MD** and is therefore equal to it; clearly also the angle **OMC** coincides with the angle **OMD** and is therefore equal to it. Hence **AB** bisects **CD** and is also perpendicular to it.

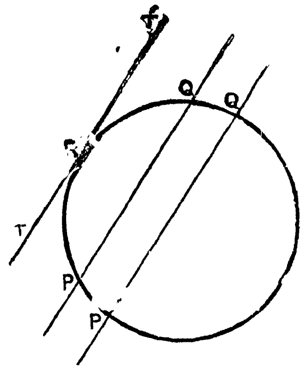
20. Any unlimited straight line that cuts the circumference of a circle is called a **secant** of a circle.

Note. Hence, a chord when produced both ways becomes a *secant*, or, in other words, a chord is that portion of a secant which lies within the circle.

21. If a secant of a circle moves so that the two points in which it cuts the circumference continually approach one another, then in the position which it ultimately takes up when the two points coincide, the secant becomes what is called a **tangent** to the circle; and it is said to *touch* or be the *tangent* to, the circle *at the point* at which those two points coincide.



Note 1. In the joining diagram, let **PQ** be a secant of the circle whose centre is **O**. Let **P** remain fixed while **Q** moves along the circumference towards **P**; then the secant **PQ** will continually change its position. If **PT** be the position which the secant assumes when **Q** coincides with **P**, then **PT** is the *tangent* to the circle at the point **P**



Note 2. Let **P, Q** be any two points on the circumference of a circle. Suppose that the secant **PQ** moves in such a manner that the points **P** and **Q** approach nearer and nearer to some point **R** on the arc **PQ**, and ultimately coincide at **R.** If **TRT′** be the position of the secant when **P** and **Q** coincide, then **TRT′** is the *tangent* to the circle at the point **R**.

Note 3. From the above it is clear that when a straight line becomes a *tangent* to a circle at any point, it does not cut the circumference at that point. Hence a tangent to a circle is *an unlimited straight line which meets the circumference but does not cut it at the point of meeting.*

Note 4. The point at which the tangent to a circle meets it, is called the **point of contact** of the tangent.

Note 5. A tangent may also be regarded as a secant cutting the circumference in two *coincident* points In other words, a tangent is a secant of which the chord-portion does not exist.

22. If the circumference of one circle passes through two near points on the circumference of another, and if one of the circles moves, while the other remains fixed, so that the two points come nearer and nearer to each other, then in its ultimate position, when the two points coincide, the moving circle is said to **touch** the other at the point of coincidence of those two points.

Note 1. In the diagram on the next page, let **O** and **O′** be the centres of the two circles which cut each other in the points **P** and **Q.** If the first circle moves, while the second remains fixed, so that the points **P** and **Q** come nearer and nearer to one another, and if

HRK be the ultimate position of the moving circle when **P** and **Q** coincide at **R**, then the circle **HRK** is said to *touch* the circle **RMN** at the point **R**.

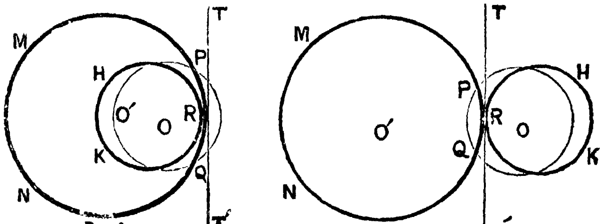


FIG. 1 FIG. 2

Note 2. Hence it is clear that two circles may be said to *touch* each other when they *meet* but *do not cut each other* at the point of meeting.

Note 3. It may also be said that one circle touches another when the circumference of the former passes through two *coincident* points on the circumference of the latter.

Note 4. In the above diagram, let **TRT′** be the ultimate position of the secant **PQ**, when **P** and **Q** coincide at **R**.

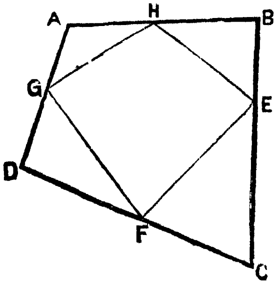
Then **TRT′** is clearly a tangent to each of the two circles, because it passes through two *coincident* points on each circumference. Hence, *when two circles touch each other, they have a common tangent at the point of contact.*

Note 5. Two circles are said to touch each other **internally**, or to have **inteanal contact** when the centres of the circles are on the *same* side of the point of contact, as in fig. (1); and they are said to touch each other **externally**, or to have **external contact**, when the centres are on opposite sides of the point of contact, as in fig. (2).

23. When all the angular points of a rectilineal figure lie on the circumference of a circle, the rectilineal figure is said to be **inscribed in the circle**, and the circle is said to be **circumscribed about the rectilineal figure.**

24. When each side of a rectilineal figure is a tangent to a circle, the rectilineal figure is said to be **circumscribed about the circle**, and the circle is said to be **inscribed in the rectilineal figure.**

25. If each vertex of one rectilineal figure lies on a side of another and there is one vertex of the first figure on each side of the second, then the first figure is said to be **inscribed in** the socond then the second is said to be **circumscribed about** the first. Thus, in the following diagram, the quadrilateral HEFG is *inscribed in* the quadrilateral ABCD, and the quadrilateral ABCD is *circumscribed about* the quadrilateral HEFG.



EXERCISE (19).

1. Let O be the centre of a circle whose radius is two inches. If from O any straight line OP be drawn equal to two inches in length, prove that the doint P is neither inside the circle nor outside it.

2. If A and B be two concentric circles such that the radius of A is smaller than that of B, prove that every point on the circumference of A is within the circle B.

3. If A and B be two concentric circles such that the radius of A is equal to that of B, prove that every point on the circumference of A is on the circumference of B.

4. Deduce, from examples 2 and 3, that if two circles cut each other they *can not have the same centre.*

5. AB is a chord of a circle of which the centre is O. If P be any point in AB and Q any point in AB produced, prove that OP is less than, and OQ is greater than, the radius of the circle.

6. When does a segment become a semi-circle ?

7. When does a sector become a segment ?

8. Every chord divides a circle into two segments ; when are the segments equal ?

9. Divide a circle into two equal sectors.

10. Prove that a semi-circle is symmetrical about a straight line drawn through the centre at right angles to the diameter.

11. Prove that a sector is symmetrical about the bisector of its angle.

12. If a chord be perpendicular to a diameter, prove that the extremities of the chord are symmetrically opposite with regard to that diameter.

13. When a secant becomes a tangent, what becomes of the extremities of the chord-portion of the secant ? Where then is the middle point of the chord-portion ?

14. Two circles that touch each other may also be regarded as cutting each other in two points. How? Prove that two circles that touch each other have a common tangent at the point of contact.

15. When is one rectilineal figure said to be inscribed in another? Show that if one or other of the two conditions were omitted from the difinition, the inner figure could not necessarily be regarded as having the same number of sides as the outer figure.

16. AB is a diameter of a circle of which the centre is O; and P is a point on the diameter between O and B. If C is any other point on the circumference, prove that PC is less than PA but greater than PB.

If D be a point on the arc AC, prove that PD is greater than PC.

17. AB is a diameter of a circle, and P is a point on BA produced. If C is any other point on the circumference, prove that PC is greater than PA but less than PB.

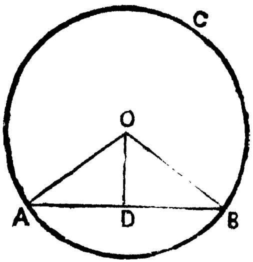
SECTION II.

THEOREMS.

Theorem 1. (Euc. III. 3.)

A straight line drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord ;

Conversely, the perpendicular to a chord from the centre bisects the chord.



Let O be the centre, and AB a chord, of the ⊙ ABC.

(i) Let D be the mid.-point of AB.

To prove that OD is ⊥ to AB.

Join OA, OB.

Proof. In the △s OAD, OBD, we have

(1) AD = DB, (*Hyp.*)

(2) OD common,

and (3) OA = OB ; (*Radii of the* ⊙)

∴ the two △s are congruent. (*Bk. I, Th. 7.*)

Hence the $\angle$ ODA = the $\angle$ ODB ;

and $\therefore$ OD is $\perp$ to AB. Q. E. D.

(ii) Let OD be $\perp$ to AB.

To prove that AD = DB.

Join OA, OB.

Proof. In the *right-angled* $\triangle$s ODA, ODB, we have

(1) OD common,

and (2) the hypotenuse OA = the hypotenuse OB ; (*Radii*)

$\therefore$ the two $\triangle$s are congruent.

Hence AD = DB. Q. E. D.

Cor. 1. *A straight line cannot cut a circle in more than two points.*

For, in the above figure, if the chord AB were to meet the circle again at some other point E, then D would be the middle point of AB as well as of AE, which is impossible.

Cor. 2. *The middle points of any system of parallel chords of a circle lie in a straight line passing through the centre.*

For, if a str. line be drawn from the centre $\perp$ to one of the chords, it will be $\perp$ to all chords, and will $\therefore$ intersect all the chords at their middle points.

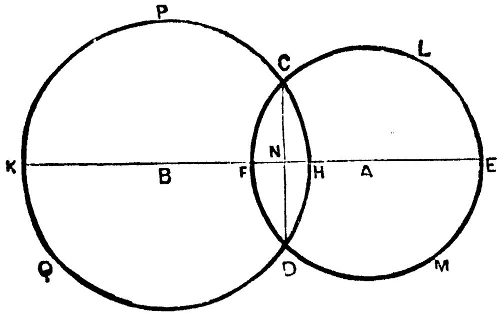
Note. 1. Of the many bisectors of **AB** there is *only* one which is *also* perpendicular to it ; and it is on *this* bisector, and on *no other*, that **O** must lie, because **O** is equi-distant from **A** and **B.** (Th. 25, Bk. I.). Hence, if it is known that **OD** is a bisector of **AB**, we at once conclude that **OD** is also perpendicular to it, because **O** can not lie on any other bisector.

Note 2. Of the many perpendiculars to AB there is *only* one which also bisects AB; and it is on *this* perpendicular, and on *no other*, that O must lie, because O is equidistant from A and B (Th. 25, Bk. I). Hence, if it is known that OD is perpendicular to AB, we at once conclude that OD also bisects it, because O can not lie on any other perpendicular.

Note 3. If a perpendicular be drawn to AB through its middle point, it *must pass through* O; because, no point that is equidistant from A and B can lie outside this perpendicular.

Theorem 2.

If two circles cut each other in two points, the line joining their centres bisects the common chord at right angles.



Let the ⊙s whose centres are A and B cut each other in the points C and D.

Join AB. Also join CD; then CD is the common chord of the two ⊙s.

To prove that AB bisects CD at right angles.

Let AB be produced both ways; and let it cut the ⊙A in E and F, and the ⊙ B in H and K.

Let N be the point in which AB and CD intersect.

Proof. Suppose the figure to be folded about the line EK, so that the upper portion of the figure falls on the side of EK remote from C.

Then, since the ⊙A is symmetrical about the diameter EF, the semi-○ᶜᵉ FLE will entirely coincide with the

semi-$\bigcirc^{ce}$ FME ; and, for a similar reason, the semi-$\bigcirc^{ce}$ HPK will entirely coincide with the semi-$\bigcirc^{ce}$ HQK.

Hence, the point C will fall upon the semi-$\bigcirc^{ce}$ FME and also upon the semi-$\bigcirc^{ce}$ HQK ; it will therefore coincide with the pt. D.

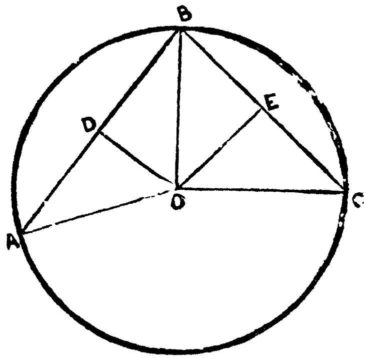
Hence NC = ND, and the $\angle$ BNC = the $\angle$ BND ; which shews that AB bisects CD at right angles. Q. E. D.

Note 1. The proposition may be also proved as follows: Through **N**, the mid -pt. of **CD**, draw the str. line **KNE** perpendicular to **CD**, Then, since the pt. **A** is equidistant from **C** and **D**, it *must lie* on the str. line **KE** (Th. 25, Bk. I.); and, for a similar reason, the pt. **B** also must lie on **KE**. Hence the str. line **AB** coincides with the str. line **KE**, and $\therefore$ it bisects **CD** at right angles.

Note 2. It may be also deduced as a corollary to Theorem 1: Let **N** be the mid.-pt. of **CD** ; join **AN** and **BN**. Then **AN** and **BN** are both perpendicular to **CD** (Th. 1); the $\angle$s **ANC**, **BNC** are therefore together = two rt. angles, and hence **AN** and **NB** are in the same str. line. Hence, the str. line **AB** bisects **CD** and is also perpendicular to it.

Theorem 3.

There is one circle, and one only, which passes through three given poins not in a straight line.



Let A, B, C be three given points which do not lie in one str. line.

To prove (i) that a circle can be drawn through A, B, C; and (ii) that there is only one circle passing through them.

Proof. (i) Let DO and EO, which are the ⊥ bisectors of AB and BC, meet at O. Join OA, OB, OC.

Then, since O is on the ⊥ bisector of AB, we have OA = OB;

and, for a similar reason, OB = OC.

$$\therefore \quad OA = OB = OC.$$

Hence the ⊙ described with centre O and radius OA will pass through all the three pts. A, B, C.

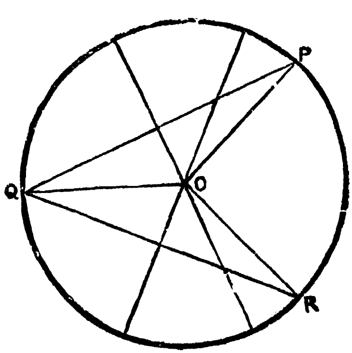
(ii) If a ⊙ be described with centre O and a radius different from OA, it will pass through *none* of the pts. A, B, C.

If a ⊙ be described with centre O′, *different from* O, it will *not* pass through *all* the three pts. A, B, C ; for, a point other than O can *not* be equidistant from those three pts.

Hence the *only* ⊙ that can pass through the three pts. A, B, and C is the ⊙ described with centre O and radius OA. Q. E. D.

Cor. 1. *One circle can not cut another in more than two points.* For, if one ⊙ has three points in common with another, then there will be *two* ⊙s. passing through the same three points, which is impossible

Cor. 2. *If from a point* O *within a circle three equal staright lines* OP, OQ, OR *can be drawn to the circumference, then* O *must be the centre of the circle.*



Since the pt. O is equidistant from the pts.. P, Q, R, it must be the pt. of intersection of the ⊥ bisectors of PQ and QR ; and there is *no other* pt. besides this which can be equidistant from those three pts. Hence the centre of the given ⊙, which certainly is equidistant from the pts. P, Q, R, on the circumference, must coincide with O.

Note 1. A circle is said to be **described** (or **circumscribed**) **about a triangle** when it passes through the vertices of the triangle. The circle described about a triangle is called the circum-circle of the triangle; and the centre and radius of this circle are respectively called the **circum-centre** and the **circum-radius** of the triangle.

Note 2. The *circum-centre* of a triangle is evidently the point of intersection of the perpendicular bisectors of any two of its sides; and the *circum-radius* is the distance of any vertex of the triangle from the circum-centre.

EXERCISE (20).

1. AB is a chord of a circle of which the centre is O. Prove that the straight line drawn from O to the middle point of AB passes through the middle point of any other chord that is parallel to AB.

2 Prove that in a circle the centre and the middle points of any number of chords that are parallel to one another lie in one straight line.

3. If two chords of a circle bisect each other, prove that their point of intersection must be the centre of the circle

Hence, prove that the diagonals of a parallelogram inscribed in a circle are equal and that the parallelogram is a rectangle.

4. If two circles, whose centres are A and B, have one point P in common, prove that they have also another point Q in common, where Q is such that PQ is bisected at right angles by AB.

5. If any number of circles pass through two given points, prove that their centres all lie in one straight line.

6. If the chords AB, AC of a circle are equally inclined to the radius to A, prove that AB = AC.

7. Prove that the circum-centre of a right-angled triangle is the mid-point of its hypotenuse.

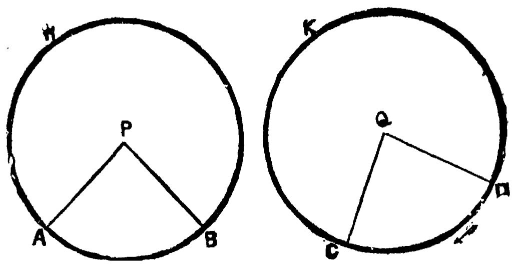
8 If two circles have two points in common, prove that any two parallel straight lines drawn through the common points and terminated by the circumferences are eqaul.

9. Construct a triangle whose sides are 2, 3 and 4 inches respectively, and find the point which is equidistant from the angular points of the triangle.

Theorem 4. (Euc. III. 26 and 27.)

In two equal circles, if there be two arcs, one in each, subtending equal angles at the centres, then these two arcs are equal.

Conversely, if there be two equal arcs, one in each, then the angles which they subtend at the centres are equal.



Let ABH and CDK be two equal ⊙s whose centres are P and Q; and let the arcs AB and CD subtend the ∠s APB and CQD respectivety at the centres.

(i) If the ∠APB be equal to the ∠CQD, to prove that the arc AB is equal to the arc CD.

Apply the ⊙ CDK to the ⊙ ABH, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the contre P, and QC may fall upon PA.

Then, since the $\angle$ CQD = the $\angle$ APB, QD will fall upon PB.

Hence, since QC = PA and QD = PB, the pts. C and D must coincide with the pts. A and B respectively.

Also, since the two ⊙s have now become concentric and they have equal radii, the two O^{ces} must coincide *everywhere*.

Hence the arc CD entirely coincides with the arc AB and is ∴ *equal* to it. Q. E. D.

(ii) **If the arc AB be equal to the arc CD, to prove that the $\angle$ APB = the $\angle$ CQD.**

Apply the ⊙ CDK to the ⊙ ABH, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the centre P, and QC may fall upon PA.

Then, since QC = PA, the pt. C must coincide with the pt. A ;

Also, since the ⊙s have now become concentric and they have equal radii, the two O^{ces} coincide *everywhere*.

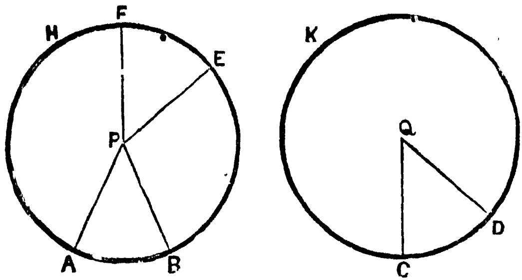
Hence, since the arc CD = the arc AB, the pt. D must coincide with the pt. B

Now, Q coinciding with P and D with B, QD coincides with PB ; and consequently the $\angle$ CQD coincides with the $\angle$ APB.

Hence, the $\angle$ CQD = the $\angle$ APB.

Q. E. D.

Cor. 1. *If in the same circle* ABH, *the arcs* AB *and* EF *subtend equal* $\angle$*s* APB, EPF *at the centre, then the arc* AB = *the arc* EF.



Let CDK be a ⊙ whose centre is Q and radius = PA, and let the ∠ CQD contained by the radii QC and QD be = the ∠ APB (or the ∠ EPF). Clearly then the arc AB = the arc EF, because each of them = the arc CD.

Cor 2. *If in the same circle* ABH *the arc* AB *be = the arc* EF, *then the* ∠*s* APB, EPF, *which these arcs subtend at the centre, are equal.*

Let CDK be a ⊙ whose centre is Q and radius = PA, and let the arc CD be = the arc AB (or the arc EF). Clearly then the ∠ APB = the ∠ EPF, because each of them = the ∠ CQD.

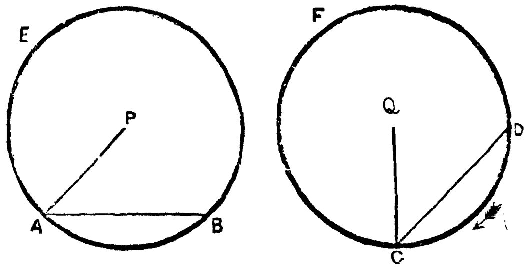
Cor. 3. *In equal circles, sectors which have equal angles are equal; also, sectors which have equal arcs are equal.*

If the ⊙ CDK be applied to the ⊙ ABH so that the ∠ CQD coincides with the ∠ APB, then the arc CD also coincides with the arc AB. Hence the sector QCD entirely coincides with the sector PAB and is ∴ equal to it. Similarly, if the arc CD be = the arc AB, it follows that the sector QCD = the sector PAB.

Theorem 5. (Euc. III. 29 and 28.)

In two equal circles, if there be two equal arcs, one in each, then the chords of these two arcs are equal;

Conversly, if there be two equal chords, one in each, then the arcs which they cut off are equal, the major arc being equal to the major arc and the minor to the minor.



Let ABE and CDF be two equal ⊙s of which the centres are P and Q.

(i) Let the arc AB be = the arc CD.

Join AB, CD.

To prove that the chord AB is equal to the chord CD.

Proof. Join PA, QC.

Apply the ⊙ CDF to the ⊙ABE, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the centre P and QC may fall upon PA.

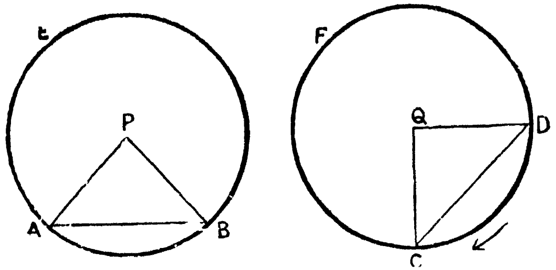
Then, since QC = PA, the pt. C must coincide with the pt. A.

Since the ⊙s have now become concentric and they have equal radii, the two ○ces coincide *everywhere.* Hence the arc CD falls upon the arc AB; and consequently, since the arcs are equal, the pt. D coincides with the pt. B.

Hence the chord CD coincides with the chord AB and is ∴ equal to it. Q. E. D.

(ii) Let the chord AB be = the chord CD.

To prove that the minor arc AB is equal to the minor arc CD, and the major arc AEB is equal to the major arc CFD.



Proof. Join PA, PB, QC, QD.

Since PA, PB, AB are respectively = QC, QD, CD, therefore the two △s PAB and QCD are congruent.

Hence the ∠ APB = the ∠ CQD.

Now, apply the ⊙ CDF to the ⊙ ABE, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the centre P, and QC may fall upon PA.

Then, since the ∠ CQD = the ∠ APB, QD will fall upon PB.

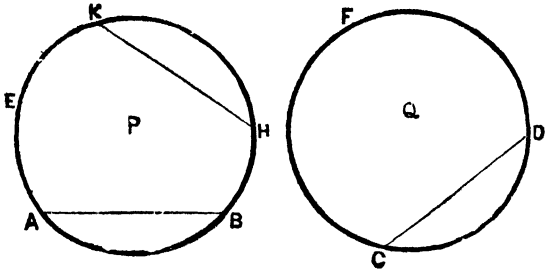
Hence, since QC = PA and QD = PB, the pts. C and D must coincide with the pts. A and B respectively.

Also, since the two ⊙s have now become concentric and they have equal radii, the two O^{ces} coincide *everywhere.*

Hence the arc CD entirely coincides with the arc AB and is ∴ equal to it.

Also the arc CFD entirely coincides with the arc AEB is ∴ equal to it. Q. E. D.

Cor. 1. *If in the same circle* ABE, *the arc* OB *be equal to the arc* HK, *then the chord* AB *is equal to the chord* HK.



Let CDF be an equal ⊙ having its centre at Q, and let the arc CD be = the arc AB (or the arc HK). Clearly then the chord AB = the chord HK, because each of them = the chord CD.

Cor. 2. *If in the same circle* ABE, *the chord* AB *be equal to the chord* HK, *then the minor arc* AB *is equal to the minor arc* HK, *and the major arc* AEB *is equal to the major arc* HEK.

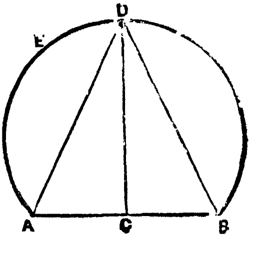
Let CDF be an equal ⊙ having its centre at Q, and let the chord CD be = the chord AB (or the chord HK).

Clearly then the minor arc AB = the minor arc HK, because each of them = the minor arc CD. Also the major arc AEB = the major arc HEK, because each of them = the major arc CFD.

Cor. 3. *In equal circles, segments which have equal arcs are equal; also, segments which have equal chords and which are both greater than or both less than a semi-circle, are equal.*

For, if the ⊙ CDF be so applied to the ⊙ ABE that the arc CD coincides with the arc AB, then the segment CD entirely coincides with the segment AB and is ∴ equal to it. Again, if the chord CD be = the chord AB, the ⊙ CDF may be so applied to the ⊙ ABE that the arc CD coincides with the arc AB and the arc CFD with the arc AEB. Hence the segment CD entirely coincides with the segment AB and is ∴ equal to it; also the segment CFD entirely coincides with the segment AEB and is ∴ equal to it.

Cor. 4. *If* C *be the middle point of the chord* AB *of an arc* AEB, *and if* CD *be drawn perpendicular to* AB *meeting the arc in* D, *then* D *is the middle point of the arc* AEB.

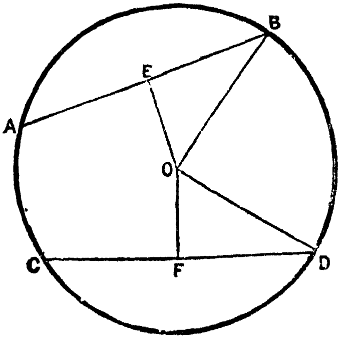


Since DC produced is a diameter of the ⊙ of which AEB is an arc, the arcs DA and DB are clearly both less than the semi-⊙^{ce} of the ⊙. Now, if DA and DB be joined, the two △s ACD and BCD are evidently congruent; whence DA = DB, and ∴ the arc DA = the arc DB.

Theorem 6. (Euc. III. 14.)

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let AB and CD be two chords of a ⊙ of which the centre is O.

Let OE, OF be the ⊥s from O upon AB and CD; then OE, OF are respectively the *distances* of AB and CD from O.

(i) If AB be = CD, **to prove that** OE = OF.

Proof. Join OB, OD.

Since OE is ⊥ to AB, ∴ AE = EB; (*Th. 1.*)

and ∴ EB = ½ AB.

Similarly, FD = ½ CD.

Hence, EB = FD.

Now, in the rt. angled △s OEB, OFD, we have

(1) the hypotenuse OB = the hypotenuse OD,

and (2) EB = FD;

∴ the two △s are congruent.

Hence, OE = OF. Q. E. D.

(ii) If OE be = OF to prove that AB = CD.

Proof. Join OB, OD.

In the rt. angled △s OEB, OFD, we have

(1) the hypotenuse OB = the hypotenuse OD,

and (2) OE = OF ;

∴ the two △s are congruent.

Hence, EB = FD. . . . . . . (*a*)

Now, OE being ⊥ to AB, E is the mid-pt. of AB ;

∴ AB = 2EB.

Similarly, CD = 2FD.

Hence, from (*a*), AB = CD. Q. E. D.

Cor. *The middle points of equal chords of a circle lie on the circumference of a concentric circle.*

For, the distances of the middle points of the chords from the centre of the ⊙ are respetively the ⊥s upon the chords from the centre, and are ∴ equal to one another,

Note. It is easy to see that $OE^2+EB^2=OF^2+FD^2$. Hence, if OE be > OF, we have EB < FD and ∴ AB < CD ; also, if OE, be < OF; we have EB > FD and ∴ AB > CD. Thus, of any two chords in a circle, that which is nearer to the centre is the greater; and conversely, that which is greater is nearer to the centre than the other. It is also clear, that OB is > EB. Hence the diameter through B and any diameter is > AB ; and similarly, the diameter through D and ∴ any diameter is > CD. Thus of all chords in a circle the greatest is any that passes through the centre.

EXERCISE (21).

1. A, B, C, are three points on the circumference of a circle of which the centre is O. If the angle AOC be three times the angle AOB, prove that the arc AC is also three times the arc AB.

2. P and Q are two points on the circumference of a circle of which the centre is O. If the $\angle POQ = 120^\circ$, prove that the arc PQ is one-third of the whole circumference, and that the chord PQ is a side of an equilateral triangle inscribed in the circle.

Hence construct an equilateral triangle with its vertices on the circumference of a circle whose radius is 2 inches.

3. If AB and CD are two diameters of a circle, perpendicular to each other, prove that the quadrilateral ACBD is both equilateral and equiangular.

4. If a regular hexagon be inscribed in a circle, prove that each side of the figure is equal to a radius of the circle.

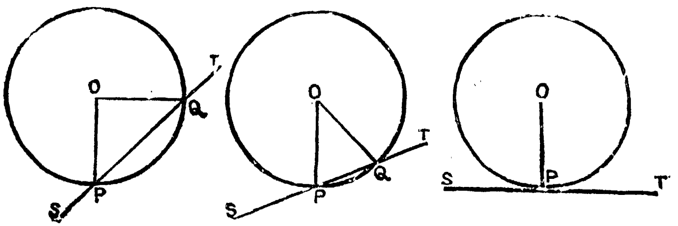
5. If two equal circles intersect so that the centre of one circle is on the circumference of the other, prove that one-third of each circumference lies within the other.

6. AB, AC, AD are three chords of a circle, taken in order, AB being a diameter. Prove that AB is greater than AC and AC is greater than AD. Prove also that the chord AD is more distant from the centre than AC.

7. O is the centre of a circle of which the radius is 2 inches, and A, B, C are three points each at a distance of one inch from O. Construct the chords that are bisected respectvely at A, B, C and prove that these chords are equal to one another. Also verify this fact by actual measurement.

Theorem 7. (Euc. III. 18.)

The tangent at any point of a circle is perpendicular to the radius through the point.



Let P be any point on the $\bigcirc^{ce}$ of a $\odot$ whose centre is O ; join OP.

To prove that the tangent at P is perpendicular to OP.

Proof. Let ST be a secant through P, cutting the $\odot$ agian at Q.

Join OQ.

Since OQ = OP, $\therefore$ the $\angle$OPQ = the $\angle$OQP.

Now, the $\angle$OPS is the supplement of the $\angle$OPQ,
and the $\angle$OQT is the supplement of the $\angle$OQP.

Hence the $\angle$OPS = $\angle$OQT.

Let the secant ST be turned about the pt. P so that the pt. Q moves along the arc QP and gradually approaches P.

Then, in every position of the secant the $\angle$ OPS = the $\angle$OQT.

Ultimately, when Q coincides with P, the secant ST becomes the tangent at P ; and the $\angle$OQT becomes the $\angle$OPT.

Hence, when ST is the tangent at P, the $\angle$OPS $\angle$OPT ;

and these are adjacent angles ;

$\therefore$ OP is $\perp$ to ST.

That is, the tangent at P is $\perp$ to OP. Q. E. D.

Cor. 1. *There cannot be more than one tangent to a circle at any given point on its circumference.*

For, if there were two tangents to a $\odot$ at any given pt. P on its $\bigcirc^{ce}$, each of them would be $\perp$ to OP, and thus there would be two $\perp$s to OP at P, which is impossible.

Cor. 2. *If through any point, on a circle a straight line be drawn perpendicular to the raduis to that point, it must be the tangent to the circle at that point.*

If PT′ be drawn $\perp$ to OP, PT′ *must* coincide in direction with PT ; for, there would otherwise be two perpendiculars to OP at P, which is impossible.

Cor. 3. *If from the point of contact of a tangent to a circle a straight line be drawn perpendicular to the tangent, it must pass through the centre of the circle.*

If PO′ be drawn $\perp$ to PT, PO′ *must* coincide in direction with PO ; for, there would otherwise be two perpendiculars to PT at P, which is impossible.

Cor. 4. *If through the centre of a circle a straight line be drawn perpendicular to a tangent to the circle, it must meet the tangent at the point of contact.*

If through O a straight line be drawn $\perp$ to the tangent SPT, it *must* coincide in direction with OP ; for, there would otherwise be two prependiculars to SPT through O, which is impossible.

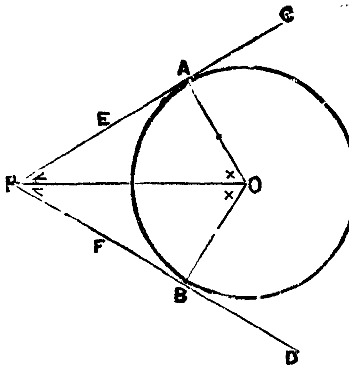
Cor. 5, *Every point on a tangent to a circle except the point of contact, is outside the circle.*

Let SPT be a tangent to a ⊙ whose centre is O, the pt. of contact being P. Then, since OP is ⊥ to ST, every other point on ST, besides P, is at a greater distance from O than OP, *i.e.*, at a greater distance from O than the radius of the ⊙ ; which proves the corollary.

Note. The proposition may also be proved as follows: The straight line drawn from the centre of a circle to the middle point of the chord-portion of a secant is always perpendicular to the secant (Th. 1). When the secant becomes a tangent, the chord-portion becomes reduced to the mere point of contact, and so the middle point of the chord-portion too becomes identical with the point of contact. Hence the straight line drawn from the centre to the point of contact of a tangent (which may be regarded as the straight line drawn from the centre to the middle point of the chord-portion of the tangent) is prependicular to the tangent.

Theorem 8.

If two tangents to a circle meet at a point, the portions of the tangents intercepted between the point and the circle are equal, they subtend equal angles at the centre of the circle, and they are also equally inclined to the straight line drawn from the point to the centre of the circle.



Let EAC and FBD be two trngents to a $\odot$ whose centre is O, the points of contact being A and B respectively. Let the tangents meet at the point P.

Join OA, OB, OP.

To prove that PA = PB.

the $\angle$ AOP = the $\angle$ BOP.

and the $\angle$ APO = the BPO.

Proof. Since OA and OB are the radii drawn to the pts. of contact, the tangents PAC and PBD are respectively $\perp$ to OA and OB. (*Th. 7.*)

Hence the $\angle$s OAP, OBP are rt. $\angle$s,

Now, in the rt.-angled Δs OAP, OBP, we have

(1) The hypotenuse OP common,

and (2) OA = OB ;

$\therefore$ The two traingles are congruent.

$$\left.\begin{array}{r}\text{Hence, } PA = PB, \\ \text{the } \angle AOP = \text{the } \angle BOP, \\ \text{and the } \angle APO = \text{the } \angle BPO.\end{array}\right\}$$ Q. E. D.

Note 1. When a tangent is drawn to a circle from a point outside it, *the portion of the tangent intercepted between the point and the circle* is often spoken of as *the tangent drawn from the point to the circle.* Hence, Theorem 8 may also be enunciated as follows : "The two tangents drawn to a circle from an external point are equal and subtend equal angles at the centre of the circle, and they are also &c."

Note 2. If AB is joined, the line AB is called the **chord of contact** of the tangents from P to the circle.

Theorem 9. (Euc. III. 11 and 12.)

If two circles touch the point of contact lies on the straight line through the centres.

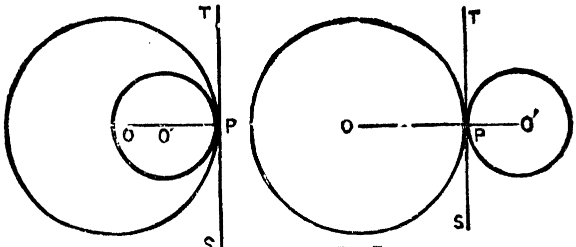


Fig. 1. Fig. 2.

Let the ⊙s whose centres are O and O′ touch at the pt. P.

To prove that O, O′ and P are in the same str. line.

Proof. The two ⊙s have a common tangent at P. (*Sec I., Art.* 22, *Note* 4.)

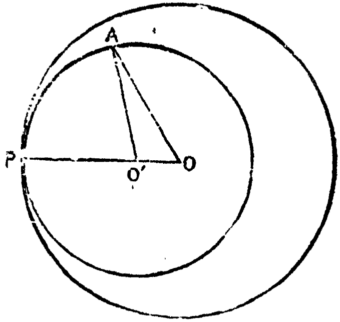
Let TPS be the common tangent; join OP, O′P.

Then OP, O′P being radii drawn to the pt. of contact, each of the ∠s TPO, TPO′ is a rt. ∠. (*Th.* 7.)

Hence in fig. (1), O′ is on the line joining PO;

and in fig, (2), O′ is on OP produced. Thus, in both the cases, the three pts. O, O′, P are in one str. line. Q. E. D.

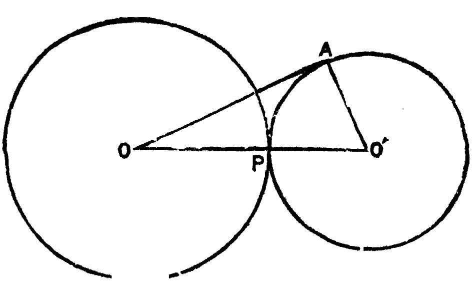
Cor. 1. *If two circles touch each other internally, every point on the circle of smaller radius, except the point of contact lies inside the other circle.*



Let the ⊙s whose centres are O and O′ touch each other internally at P, the second ⊙ being of smaller radius. Then O, O′, P are in one str. line and OP > O′P. Take *any* pt. A, *other than* the pt. P, on the ⊙ of smaller radius: join OA, O′A. Now, OA is less than the sum of OO′ and O′A, and is ∴ less than OP. Thus the distance of A from the centre of the first ⊙ is less than its radius and ∴ A is *inside* the first ⊙; which proves the corollary.

Cor. 2. *If two circles touch each other externally, every point on either circle, except the point of contact, lies outside the other.*

Let the ⊙s whose centres are O and O′ touch each other externally at P Then O, O′, P are in one str. line. Take *any* pt. A, *other than* the pt. P, on the second ⊙ ; join OA, O′A. Now, OA + O′A > OO′ but O′A = O′P ; ∴ OA > OP. Thus the distance of A from the centre of the first ⊙ is greater than its radius, and ∴ A is *out side* the first ⊙ ; which proves that *every* pt. on the second ⊙,



except P, is outside the first. Similarly, every pt. on the first ⊙; except P, is outside the second.

Note 1. The main proposition may also be proved as follows: If two circles cut each other in two points, their centres and the mid-pt. of the common chord lie in one str. line: (Th. 2.). When the circles touch, the two pts. of intersection coincide; the common chord, therefore, becomes reduced to the mere pt. of contact, and consequently the mid-pt. of the chord also becomes identical with the point of contact. Hence, when the circles touch, their centres and the point of contact must lie in one str. line.

Note 2. *If two circles have a common point on the line joining their centres, the circles touch each other at that point.*

For, if the circles intersect in two distinct points, the line joining the centres passes through the mid-pt. of the common chord and never through one or other of the pts. of intersection.

Note 3. *If two circles have the same tangent at a common point they touch each other at that point.*

For, if the circles intersect in two distinct pts., they cannot have common tangent at either of those pts., they can only have a common *secant* passing through the two pts. of intersection.

EXERCISE (22).

1. If two circles are concentric, prove that any chord of the outer circle which touches the inner is bisected at the point of contact.

2. If any number of equal circles touch a given straight line on the same side of it, prove that their centres all lie in one straight line.

3. If P be a point from which tangents are drawn to a circle whose centre is O, prove that OP is the perpendicular bisector of the chord of contact.

4. If any number of circles pass through the same point and touch one another at that point, prove that their centres all lie in one str. line.

5. If any number of circles touch each of two intersecting straight lines, prove that their centres all lie in one straight line.

6. If a parallelogram be described about a circle, prove that the sum of one pair of its opposite sides is equal to that of the other pair. Hence prove that any two adjacent sides of the parallelogram are equal.

7. If a quadrilateral be such that the sum of one pair of its opposite sides is equal to that of the other pair, prove that a circle which touches any three sides of this quadrilateral must also touch the fourth.

8. If three equal circles touch one another at P, Q, R, prove that the tangents at P, Q, R meet at a point.

Theorem 10. (Euc. III. 20.)

The angle which an arc of a circle subtends at the centre is double that which is subtends at any point on the remaining part of the circumference.

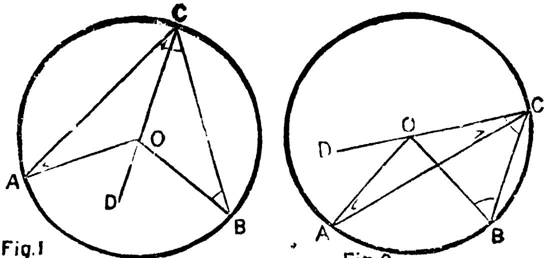


Fig. 1 Fig. 2

Let O be the centre of a ⊙ of which AB is an arc, and C is a pt. on the remaining part of the $\bigcirc^{ce}$.

Join OA, OB, CA, CB.

To prove that the ∠ AOB = twice the ∠ ACB.

Proof. Join CO and produce it to any pt D.

Since OB = OC, ∴ the ∠ OCB = the ∠ OBC.

Hence, the ∠ DOB = the ∠ OCB + the ∠ OBC
= twice the ∠ OCB.

Similarly, the ∠ AOD = twice the ∠ OCA.

Hence, in fig. (1),

the ∠ DOB + the ∠ AOD = twice the ∠ OCB + twice the ∠ OCA,

i.e., the ∠ AOB = twice the ∠ ACB,

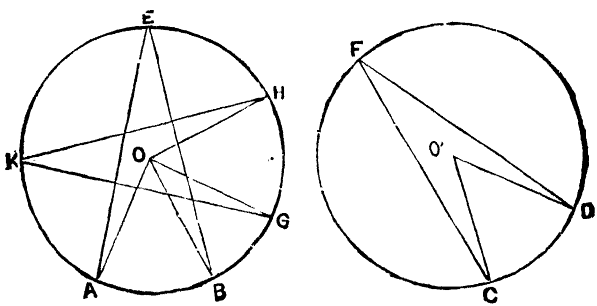
and in fig. (2),

the ∠ DOB *minus* the ∠ AOD = twice the ∠ OCB *minus* twice the ∠ OCA,

i.e., the ∠ AOB = twice the ∠ ACB. Q. E. D.

Note. If the arc **AB** be equal to the semi-circumference of the circle, the $\angle$**AOB** becomes a *straight-angle* ; and if the arc **AB** be greater than the semi-circumference, then the $\angle$**AOB** becomes a *reflex* angle. The same proof will hold for these two cases as for fig. (1).

Cor. 1. *In equal circles (or in the same circle), arcs which subtend equal angles at the circumferences (or circumference) are equal.*



Let O, O′ be the centres of two equal ⊙s. If the arcs AB and CD subtend equal $\angle$s AEB, CFD at the ○^{ces}, then they also subtend equal $\angle$s at the centres, (because the $\angle$s AOB, CO′D are respectively doubles of the $\angle$s AEB, CFD) ; and ∴ the arc AB = the arc CD. (*Th. 4.*)

If in the same ⊙, the arcs AB and GH subtend equal $\angle$s AEB, GKH at the ○^{ce}, then they also subtend equal $\angle$s at the centre, (for the $\angle$s AOB, GOH are respectively doubles of the $\angle$s AEB, GKH) ; and ∴ the arc AB = the arc GH. (*Th. 4. Cor. 1.*)

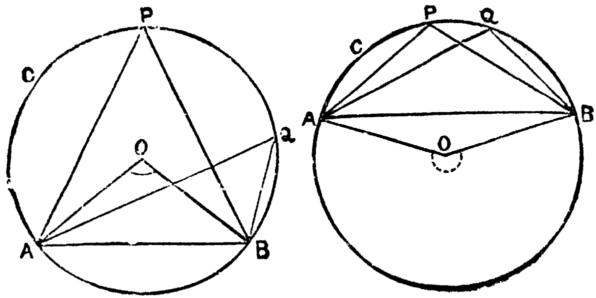
Cor. 2. *In equal circles (or in the same circle), angles at the circumferences (or circumference) which stand on equal arcs are equal.*

In the preceding diagram, if the arc AB be = the arc CD, then the ∠AOB = the ∠CO′D, (*Th. 4.*); and ∴ the ∠AEB = the ∠CFD, because these ∠s are respectively halves of the ∠s AOB, CO′D.

Again, if the arc AB be = the arc GH, then the ∠AOB = the ∠GOH, (*Th. 4, Cor. 2.*); and ∴ the ∠AEB = the ∠GKH, because these ∠s are respectively halves of the ∠s AOB, GOH.

Theorem 11. (Euc. III. 21.)

Angles in the same segment of a circle are equal.



Let APB be any ∠ in the segment ACB of the ⊙ whose centre is O. Let AQB be *any other* ∠ in the same segment.

To prove that the ∠ APB = the ∠ AQB.

Proof. Join OA, OB.

The arc AB of the ⊙ subtends the ∠ AOB at the centre, and the ∠ APB at the pt. P on the remaining part of the Oce ;

∴ the ∠ APB = half the ∠ AOB. (*Th. 10.*)

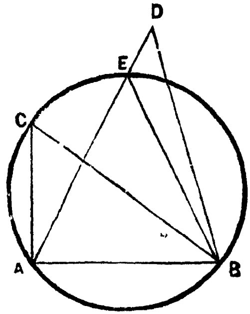
Similarly, the ∠ AQB = half the ∠ AOB.

Hence, the ∠ APB = the ∠ AQB. Q. E. D.

Note. In the second circle of the preceding figure, the segment **ACB** being less than a semi-circle, the angle **AOB** is a *reflex* angle.

Theorem 12.

If the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.



Let C and D be two pts. on the same side of the str. line AB, and let the ∠ ACB be = the ∠ ADB.

To prove that the four points A, B, C, D lie on the circumference of a circle.

Proof. A ⊙ will pass through *any three* of the four pts. (*Th. 3.*)

Hence, all that is required is to prove that the ⊙ passing through three of the pts. will also pass through the remaining pt.

If the ⊙ passing through A, B, C do not also pass through D, then it will cut AD or AD produced at some pt. E.

Join EB.

Now, the ∠s ACB and AEB being in the same segment,

the $\angle$ ACB = the $\angle$ AEB.

But the $\angle$ ACB = the $\angle$ ADB ; (*Hyp.*)

$\therefore$ the $\angle$ AEB = the $\angle$ ADB, which is impossible.

Hence the $\odot$ passing through A, B, C must also pass through D. Q. E. D.

Cor. *Triangles standing on the same base, and on the same side of it, with equal vertical angles, have their vertices on the circumference of a circle of which the given base is a chord.*

Let AB be the common base, and C, D, E, F, &c., the vertices of the $\triangle$s ; then, since the $\angle$ ACB = the $\angle$ ADB, the $\odot$ passing through A, B, C also passes through D. Similarly, the $\odot$ passing through A, B, C also passes through E ; and so on. Thus the pts. D, E, F, &c., all lie on the $\odot$ that passes through the pts. A, B, C ; which proves the corollary.

Note. The fore-going corollary and Theorem 11 are converse propositions. For, the corollary may be stated in a slightly altered form, thus :—

If triangles standing on the same base and on the same side of it, have equal vertical angles, then their vertices lie on the arc of a segment of which the given base is the chord ; whilst Theorem 11 may also be enunciated as follows :—

If triangles standing on the same base, and on the same side of it, have their vertices on the arc of a segment of which the given base is the chord, their vertical angles are equal.

EXERCISE (23).

1. ABCD is a circle of which the centre is O. If the sum of the angles ADB, BDC be equal to one right angle, prove that the points A, O, C lie in one straight line.

2. If in a circle two arcs be such that they subtend complementary angles at the circumference, prove that the two arcs are together equal to half the circumference of the circle.

3. If the chords AB, CD of a circle intersect at right angles, prove that the sum of the arcs AC, BD is equal to half the circumference of the circle.

4. P is any point within a circle of which the centre is O. If AB, CD be any two chords passing through P, prove that the angle APC is equal to half the sum of the angles AOC, BOD.

5. AB, CD are two chords of a circle such that when produced they meet at P. If O is the centre of the circle, prove that the angle APC, is equal to half the difference between the angles AOC, BOD.

6. AB is the common chord of two equal circles. If any straight line drawn through B meet the circumferences in P and Q, prove that the triangle PAQ is isosceles.

7. Two given circles intersect in the points A and B. If two straight lines pass through B of which one meets the circumferences in P and Q and the other in M and N, prove that PQ and MN subtend equal angles at A.

8. ABC is a triangle and D, E, F are the middle points of the sides opposite to A, B, C. If M be the foot

of the perpendicular from A upon BC, prove that each of the angles FDE, FME is equal to the angle BAC ; and hence prove that a circle will pass through the four points D, E, F, M.

9. ABCD is a circle. The bisectors of the angles CAB, CBA meet at P, and the bisectors of the angles DAB, DBA meet at Q. Prove that the four points A, Q, P, B lie on the circumference of the same circle.

Theorem 13. (Euc. III. 31.)

The angle in a semi-circle is a right angle; the angle in a segment greater than a semi-circle is less than a right angle; and the angle in a segment less than a semi-circle is greater than a right angle.

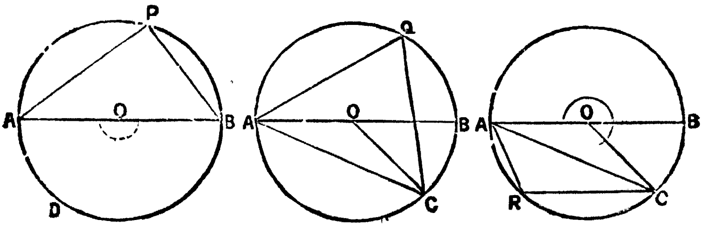


Fig 1 Fig 2 Fig 3

Let AB be a diameter of a ⊙ of which the centre is O.

(i) Let APB be any ∠ in the semi-⊙ APB, as in fig. (i).

To prove that the ∠ APB is a right angle.

Proof. The semi-$\bigcirc^{ce}$ ADB subtends the *straight angle* AOB at the centre. and the ∠ APB at the pt. P on the remaining part of the $\bigcirc^{ce}$;

∴ the ∠ APB = half the str. ∠ APB (*Th. 10.*)
= half of two rt. ∠ s.
= one right angle. Q. E. D.

(ii) Let AC be a chord of the ⊙, as in fig. (2).

Then the segment ABC is > a semi-⊙; let AQC be any ∠ in this segment.

To prove that the ∠ AQC is less than a right angle.

Proof. Join OC.

The $\angle AQC$ = half the $\angle AOC$. (*Th. 10.*)

But the $\angle AOC$ is $<$ two rt. $\angle$s ;

$\therefore$ the $\angle AQC$ is $<$ one right angle. Q. E. D.

(iii) Let AC be a chord of the $\odot$, as in fig. (3).

Then the segment ARC is $<$ a semi-$\odot$;

let ARC be any $\angle$ in this segment.

To prove that the $\angle ARC$ is greater than a rt. angle.

Proof. Join OC.

The arc ABC subtends the *reflex* $\angle AOC$ at the centre, and $\angle ARC$ at the pt. R on the remaining part of the $\odot^{ce}$,

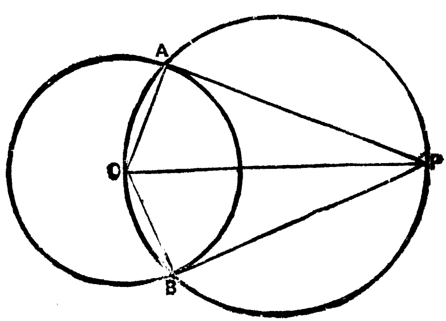
$\therefore$ the $\angle ARC$ = half the reflex $\angle AOC$. (*Th. 10.*)

But the *reflex* $\angle AOC$ is $>$ two rt. $\angle$s ;

$\therefore$ the $\angle ARC$ is $>$ one right angle. Q. E. D.

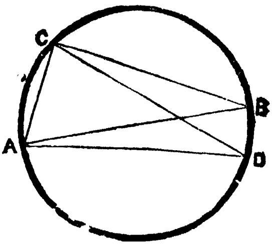
Note. From the first part of this proposition we deduce the construction of a tangent to a circle from any point outside it.

Let **P** be a point outside the $\odot$ whose centre is O. Join **OP**, and let the $\odot$ on **OP** as diameter cut the given circle at **A** and **B**; join **OA**, **OB**, **AP**, **BP**. Then the $\angle$**OAP** being in a semi-$\odot$, is a right angle. Thus **AP** is $\perp$ to the radius O



and $\therefore$ **AP** is a tangent to the given $\odot$. (*Th. 7, Cor. 2.*)

Similarly **BP** is a tangent to the given $\odot$.

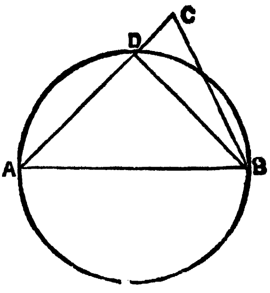


Cor. 1. *If a chord* AB *of a circle subtends a right angle at the point* C *on circumference, then* AB *is a diameter of the circle.*

If AB be not a diameter of the ⊙, let a different chord AD be the diameter through the pt. A, as in the accompanying diagram.

Join CD. Then the ∠ ACD, being in a semi-⊙, is a rt. ∠, and ∴ equal to the ∠ ACB, which is impossible. Hence AD is not a diameter; and similarly, no other chord through A, different from AB, can be a diameter. Hence AB is the diameter through A.

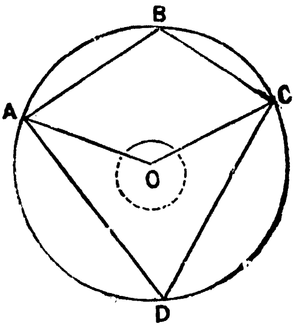
Cor. 2. *If a straight line* AB *subtends a right angle at any point* C, *then the circle described on* AB *as diameter passes through* C.



If the ⊙ does not pass through C, let it cut AC or AC produced at some pt. D, as in the accompanying diagram. Join DB. Then the ∠ ADB, being in semi-⊙, is a rt. ∠, and ∴ equal to the ∠ ACB, which is impossible. Hence the ⊙ must pass through C.

Theorem 14. (Euc. III. 22.)

The opposite angles of any quadrilateral inscribed in a circle are supplementary.



Let ABCD be a quadrilateral inscribed in a ⊙ whose centre is O.

To prove that (i) the ∠ABC + the ∠ADC = two right angles.

and (ii) the ∠BAD + the ∠BCD = twe right angles.

Proof. Join OA, OC.

The arc ABC subtends the ∠AOC at the centre, and the ∠ADC at the pt. D on the remaining part of the ○^{ce};

∴ the ∠ADC = half the ∠AOC. (*Th. 10.*)

Again, the arc ADC subtends the *reflex* ∠AOC at the centre, and the ∠ABC at the pt. B on the remaining part of the ○^{ce}.

∴ the ∠ABC = half the *reflex* ∠AOC. (*Th. 10.*)

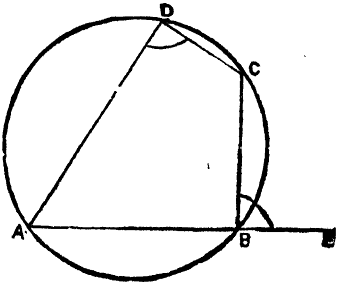
Hence, the ∠ADC + the ∠ABC = half the sum of the ∠AOC and the *reflex* ∠AOC.

= half of four right angles

= two right angles.

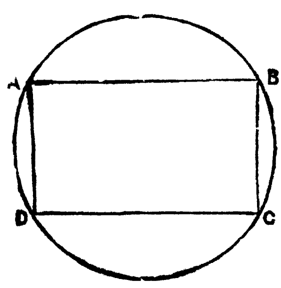
Similarly, the $\angle$ BAD + the $\angle$ BCD = two right angles.

Cor. 1. *If the side* AB *of a quadrilateral* ABCD *inscribed in a circle be produced to* E, *the angle* EBC *is equal to the angle* ADC.



The $\angle$ EBC is supplementary to the $\angle$ ABC; also the $\angle$ ADC is supplementary to the $\angle$ ABC. Hence, the $\angle$ EBC = the $\angle$ ADC.

Cor. 2. *If a circle passes through the angular points of a parallelogram, the parallelogram must be a rectangle.*



Let a ⊙ pass through the angular pts. of the par^{m}. ABCD. The $\angle$ A is = the $\angle$ C

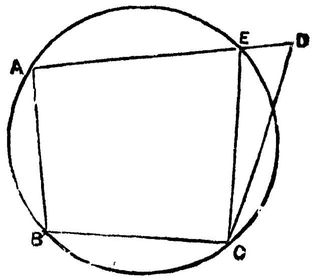
(*opp.* $\angle$*s of a par^{m}.*);

also, the sum of the $\angle$s A and C is two rt. $\angle$s, because the quadl. ABCD is inscribed in a ⊙. Hence, each of the $\angle$s A and C is a rt. $\angle$, which proves the corollary.

Note. A quadrilateral is said to be **cyclic** when a circle can pass through its angular points. Four or more points are said to be **concyclic** when a circle can pass through them all. Three or more points are said to be **collinear** when they lie in one straight line.

Theorem 15.

If two opposite angles of a quadrilateral are supplementary, a circle can be described through its angular points.



Let the $\angle$s B and D of the quadl. ABCD be together equal to two rt. $\angle$s.

To prove that the four points A, B, C, D are concyclic.

Proof. A $\odot$ will pass through the pts. A, B, C. If this $\odot$ does not pass through D, let it cut AD or AD produced at some pt. E. Join EC.

Now, the quadl. ABCE being cyclic, the $\angle$ AEC is the supplement of the $\angle$ ABC;

and the $\angle$ ADC is also the supplement of the $\angle$ ABC. (*Hyp.*)

$\therefore$ the $\angle$ AEC = the $\angle$ ADC, which is impossible.

Hence the $\odot$ which passes through A, B, C must also pass through D. Q. E. D.

Note. It may be observed that if the $\angle$s **B** and **D** are supplementary the $\angle$s **A** and **C** are also supplementary, because the four angles of a quadrilateral are together equal to four right angles.

Cor. *If, when the side* **AB** *of a quadrilateral* **ABCD** *is produced to* E *the angle* **EBC** *be equal to the angle* **ADC**, *then the four points* **A, B, C, D** *are concyclic.*

EXERCISE (24).

1. Two circles intersect at A and B. If AP and AQ are diameters, prove that the three points P, B, Q are collinear.

2. ABC is a triangle, and the circle described on AB as diameter cuts BC at D. Prove that the circle described on AC as diameter also passes through D.

3. Circles are described on the sides of a quadrilateral as diameters. Prove that the common chord of two of these circles which are adjacent is parallel to the common chord of the other two.

4. If a triangle is inscribed in a circle, prove that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

5. If a quadrilateral is inscribed in a circle, prove that the sum of the angles in the four segments exterior to the quadrilateral is equal to six right angles.

6. ABC is a triangle, and D is the middle point of BC. If BP, CQ be drawn perpendicular to AC, AB respectively, prove that the perpendicular bisector of PQ will pass through D. Prove also that the triangles ABC and APQ are equiangular.

7. In the figure of the preceding example prove that the angle BCQ is equal to the angle BPQ.

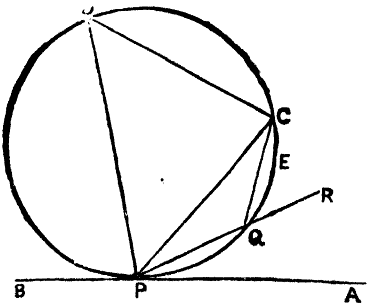
8. Prove that the straight lines, which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet on the circumference of the circle.

9. A quadrilateral is inscribed in a circle. If the bisectors of two of its opposite angles meet the circle in P and Q, prove that PQ is a diameter.

10. Prove that the four straight lines, bisecting the angles of any quadrilateral, form a cyclic quadrilateral,

Theorem 16. (Euc. III. 32.)

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are respectively equal to the angles in the alternate segments.



Let the straight line AB touch the ⊙PCD at the point P; and let PC be a chord drawn through P, dividing the ⊙ into the segments CDP and CEP.

To prove that the ∠CPA = **the angle in the segment** CDP, **and the** ∠CPB = **the angle in the segment** CEP.

Proof. In the arc CEP take a point Q close to P; join PQ and produce it to R. Join CQ, DC, DP.

Now, the quadl. PQCD being cyclic, the ∠CQR = the ∠CPD; (*Th. 14, Cor. 1.*)

and this equality holds good however close to P the point Q may be taken.

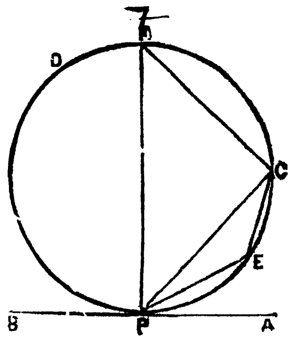
Suppose the secant PR to be turned about P, so that Q moves towards P; then, ultimately, when Q coincides with P, the ∠CQR becomes the ∠CPA.

Hence the ∠CPA = the ∠CDP.

By similar construction and reasoning, the ∠CPB = the angle in the segment CEP. Q. E. D.

Alternative Proof.

Draw the diameter PF through P. Join FC, CE, EP.



The ∠PCF, being in a semi-⊙, is a rt. ∠:

Hence, the ∠CFP + the ∠FPC = one rt. ∠;

also, since FP passes through the centre, ∴ the ∠FPA is a rt. angle. (*Th. 7.*)

∴ the ∠FPA = the ∠CFP + the ∠FPC.

Hence, taking away the common ∠FPC, the ∠CPA = the ∠CFP.

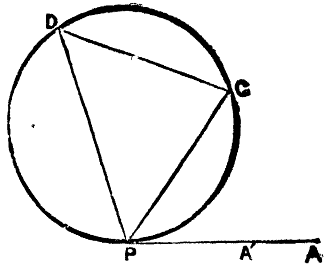
Again, the quadl. PECF being cyclic, the ∠CEP is the supplement of the ∠CFP;

also, the ∠CPB is the supplement of the ∠CPA.

But, supplements of equal ∠s are equal;

∴ the ∠CPB = the ∠CEP. Q. E. D.

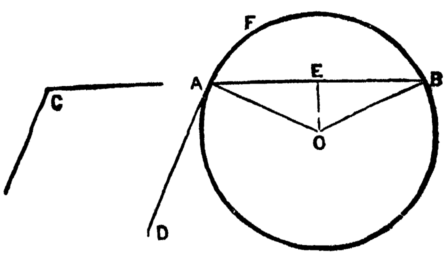
Cor. *If* PDC *is a triangle, and if through* P *a straight line* PA *be drawn on the side of* PC *remote from* D *such that the angle* CPA *is equal to the angle* CDP, *then* PA *is the tangent at* P *to the circum-circle of the triangle* PDC.



If PA′ be the tangent to the circum-circle at P, then the ∠ CPA′ = the ∠ CDP; but the ∠ CDP = the ∠ CPA. (*Hyp.*)

∴ the ∠ CPA′ = the ∠ CPA; hence PA′ and PA coincide, *i.e.*, PA is the tangent at P to the circle PDC.

Note. *We can apply this proposition to construct on a given straight line* AB *a segment of a circle containing an angle equal to a given angle* C.



Make the ∠BAD = the ∠C. Let the perpendicular bisector of AB and the perpendicular to AD through A meet at O. Join OB. Since O is on the ⊥ bisector of AB, ∴ OA = OB.

Hence the ⊙ described with O as centre and OA as radius will pass through B.

Now, since AD is ⊥ to the radius OA,

∴ AD touches the ⊙ at A. (*Th.* 7, *Cor.* 2.)

Hence the ∠ in the segment AFB = the ∠BAD = the ∠C. Thus, AFB is the required segment.

Theorem 17. (Euc. III. 35 and 36.)

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

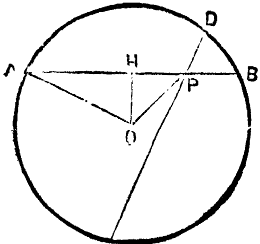


Fig. 1

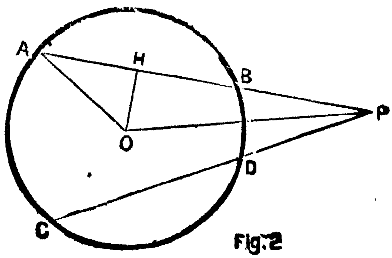


Fig. 2

Let the chords AB, CD of the ⊙ whose centre is O intersect inside or outside the ⊙ at P.

To prove that AP.PB = CP.PD.

Proof. Let OH be the ⊥ from O upon AB.
Join OP, OA.

Since OH is ⊥ to AB, ∴ AH = HB. (*Th. 1.*)

(i) When P is inside the ⊙, as in fig. (1),

$$\begin{aligned} OA^2 - OP^2 &= (AH^2 + HO^2) - (PH^2 + HO^2) \\ &= AH^2 - PH^2 \\ &= (AH + PH)(AH - PH) \quad (\textit{Th. 12, Bk. II.}) \\ &= (AH + PH)(BH - PH) \\ &= AP.\ PB; \end{aligned}$$

and similarly, $OC^2 - OP^2 = CP.PD$.

Hence, since $OA^2 = OC^2$, we have AP.PB = CP.PD.

(ii) When P is outside the ⊙, as in fig. (2),

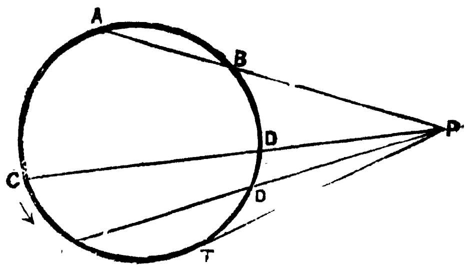
$$OP^2 - OA^2 = (PH^2 + HO^2) - (AH^2 + HO^2)$$
$$= PH^2 - AH^2$$
$$= (PH + AH)(PH - AH) \qquad (Th.\ 12,\ Bk.\ II.)$$
$$= (PH + AH)(PH - HB)$$
$$= AP.PB;$$

and similarly, $OP^2 - OC^2 = CP.PD$.

Hence, since $OA^2 = OC^2$, we have

$$AP.PB = PC.PD. \qquad \text{Q. E. D.}$$

Cor. 1. *If from any point* P *without a circle two straight lines be drawn, one of which cuts the circle in* B *and* A, *and the other touches it at* T, *then*, $PA.PB = PT^2$.

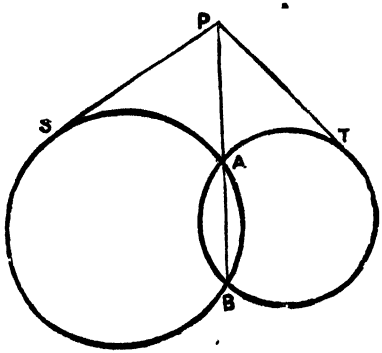


Let the secant PDC cut the ⊙ in D and C. Suppose PC to be turned about P in the direction of the arrow-head, so that the pts. C and D continually approach each other.

In every position of the secant PC, we have $PA.PB = PC.PD$; hence, ultimately, when the pts. C and D coincide at T and the secant PDC becomes the tangent PT, we must have $PA.PB = PT^2$.

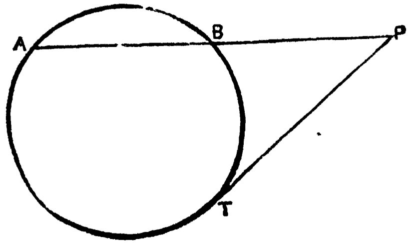
Note. Hence it follows that *if two circles cut each other in two*

points, the tangents drawn to the circles from any point on the common chord produced are equal.



The above figure explains itself. We have $PS^2 = PT^2$, because each of them $= PA . PB$; $\therefore$ $PS = PT$.

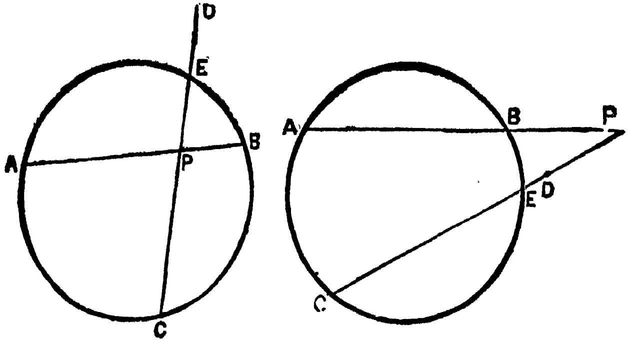
Cor. 2. *If from any point* P *without a circle two straight lines be drawn, one cutting the circle in* B *and* A *and the other meeting it at* T, *so that* $PA . PB = PT^2$, *then* PT *touches the circle at* T.



If PT do not touch the ⊙ at T, let PT produced cut the ⊙ again in some other pt. S. Then, $PT . PS = PA . PB$, and $\therefore$ $PT . PS = PT^2$, which is impossible.

Hence PT touches the ⊙ at T.

Cor. 3. *If two straight lines* AB *and* CD, *or both of them produced, intersect at* P, *so that* AP. PB = CP. PD, *then the four points* A, B, C, D *are concyclic.*



A ⊙ will pass through the pts. A, B, C. If this ⊙ do not pass through D, let it cut CD or CD produced at E. Then we must have AP. PB = CP. PE, and ∴ CP. PD = CP. PE, which is impossible. Hence the ⊙ that passes through A, B, C must also pass through D.

EXERCISE (25).

1. P is the middle point of an arc APB of a circle. Prove that the tangent at P is parallel to the chord AB.

2. If an equilateral triangle be inscribed in a circle, prove that the tangents at its angular points will form another equilateral triangle.

3. Two circles intersect at A and B; and through P, any point on the circumference of one of them, str. lines PAC, PBD are drawn to cut the other ⊙ at C and D. Prove that CD is ∥ to the tangent at P.

4. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse. Prove that the square on CD is equal to the rectangle AD. DB.

5. Construct a triangle, having given the base, the vertical angle, and the length of the str. line drawn from the vertex to the middle pt. of the base.

6. If two circles intersect, prove that their common chord produced bisects their common tangent.

7. Two ⊙s touch each other externally. Prove that any str. line drawn through the pt. of contact cuts off similar segments from the two ⊙s.

8. ABC is a triangle, and DE is drawn parallel to BC cutting the sides in D and E. Prove that the circles circumscribed about the triangles ADE and ABC have a common tangent at A.

9. ABC is a triangle; and BE, CF are the perpendiculars from B and C upon the opposite sides, intersecting each other at O. Prove that (i) BO.OE = CO.OF, (ii) AB.AF = AC.AE.

10. AB is the common chord of two given circles; and through any point in AB two chords are drawn, one in each circle. Prove that the four extremities of these chords all lie on the circumference of a circle.

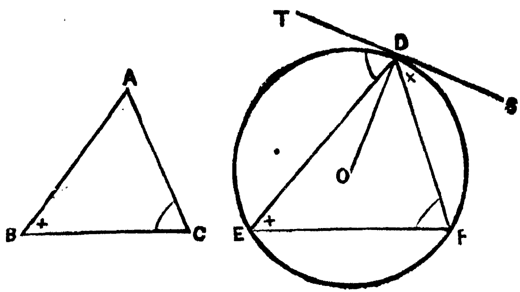
11. ABC is triangle right-angled at C, and CD is the perpendicular from C upon the hypotenuse. Prove that $AB . AD = AC^2$.

SECTION III.

PROBLEMS.

Problem 1. (EUC. IV. 2.)

In a given circle to inscribe a triangle equiangular to a given triangle.



Let O be the centre of the given ⊙, and ABC the given △.

It is required to inscribe in the circle (O) a triangle equiangular to the triangle ABC.

Cons. Take any radius OD, and through D draw TS ⊥ to OD; then TS is the tangent at D.

Draw the chord DE making the ∠TDE = the ∠ACB; and draw the chord DF making the ∠SDF = the ∠ABC.

Join EF.

Then DEF is the reqd. △.

Proof. Since TS touches the ⊙ at D,

∴ the ∠TDE = the ∠DFE, in the alt. segment;
and the ∠SDF = the ∠DEF, in the alt. segment.

Hence the $\angle$ DFE = the $\angle$ ACB,
and the $\angle$ DEF = the $\angle$ ABC ;

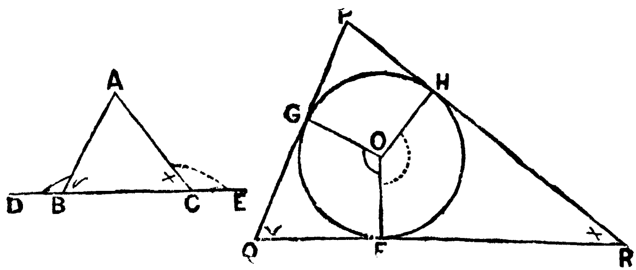
$\therefore$ the remaining $\angle$ EDF = the remaining $\angle$ BAC.

Thus, the $\triangle$DEF inscribed in the given $\odot$ is equiangular to the $\triangle$ABC. Q. E. F.

Note To inscribe an equilateral triangle in a given circle, all that we have to do is to construct an equilateral triangle outside the circle and then inscribe in the circle a triangle equiangular to the triangle constructed, in the manner indicated above. In this case therefore, each of the angles TDE and SDF would be equal to an angle of an equilateral triangle, *i.e.* equal to one-third of two right angles.

Problem 2. (Euc. IV. 3.)

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let O be the centre of the given ⊙, and ABC the given △.

It is required to circumscribe about the circle (O) a trirngle equiangular to the triangle ABC.

Cons. Produce BC both ways to D and E.

Take any radius OF of the ⊙.

Draw the radius OG making the ∠FOG = the ∠ABD; and draw the radius OH making the ∠FOH = the ∠ACE.

Draw tangents to the ⊙ at the pts. F, G, H, forming the △PQR.

Then PQR is the reqd. △.

Proof. The ∠s OFQ and OGQ being rt. ∠s, the remaining ∠s FOG and GQF of the quadl. OFQG are supplementary.

Now, the ∠ABC is the supplement of the ∠ABD, and the ∠GQF is the supplement of the ∠FOG;

but the $\angle$ ABD = the $\angle$ FOG,

$\therefore$ the $\angle$ ABC = the $\angle$ PQR.

Similarly, the $\angle$ ACB = the $\angle$ PRQ.

$\therefore$ the remaining $\angle$ QPR = the remaining $\angle$ BAC.

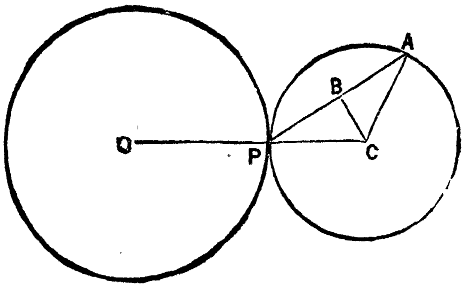
Thus, the $\triangle$PQR circumscribed about the given $\odot$ is equiangular to the $\triangle$ABC. Q. E. F.

Note 1. To circumscribe an equilateral triangle about a given circle, all that we have to do is to construct an equilateral triangle outside the circle and then circumscribe about the circle a triangle equiangular to the triangle constructed, in the manner indicated above. In this case, therefore, each of the angles **FOG**, **FOH** would be equal to the supplement of an angle of an equilateral triangle, *i.e.*, equal to two-thirds of two right angles.

Note 2. An equilateral triangle is evidently a *regular* figure of three sides; for an equilateral $\triangle$ is also equiangular. Hence, from Note to Problem 1 and Note 1 above, we know how to construct a regular figure of three sides in or about a given $\odot$.

Problem 3.

To construct a circle passing through a given point and touching a given circle at a given point.



Let A be the given pt. outside the given ⊙ whose centre is O, and let P be the given pt. on the O^{ce} of the given ⊙.

It is required to describe a circle passing through A and touching the given circle at P.

Cons. Join AP; and draw the ⊥ bisector of AP (*Prob. 2, Bk. I.*), meeting OP produced in C.

Then the ⊙ described with C as centre and CP as radius, will be the reqd. ⊙.

Proof. Join CA.

BC being the ⊥ bisector of AP, CA = CP,

and ∴ the ⊙ described with C as centre and CP as radius, will also pass through A.

Also, since the two ⊙s meet at P which is a pt. on the line joining their centres, ∴ the ⊙s touch each other at P.

(*Th. 9, Note 2.*)

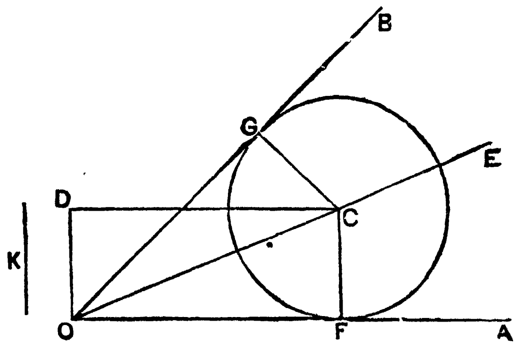
Hence, the ⊙ described with C as centre and CP as radius is the reqd. ⊙.

Note. The solution of the present problem is discovered in the following manner :—*Assume* a circle to be described passing through **A** and also touching the given circle at **P**, and let **C** be its centre. Then, since the circles touch each other at **P**, **OP** produced must pass through **C**; also, since **CA=CP**, **C** must be on the perpendicular bisector of **AP**. Hence **C** is the pt. where the perpendicular bisector of **AP** meets **OP** produced.

The process by which the solution of a problem is discovered is called **analysis**; whilst the opposite process, that of giving the necessary construction and proving its correctness, is called **synthesis**.

Problem 4.

To construct a circle, with a given radius, touching two given straight lines.



Let OA, OB be the two given str. lines intersecting at O, and let K be the given radius.

It is required to describe a circle which shall touch OA and OB, and have a radius equal to K.

Cons. Draw OE, the bisector of the $\angle$ AOB.

(*Prob. 1, Bk. 1.*)

Draw OD $\perp$ to OA, making it = K.

Through D, draw a str, line $\parallel$ to OA meeting OE in C.

Draw CF, CG $\perp$ to OA, OB respectively. Then the $\odot$ described with C as centre and CF as radius will be the reqd. $\odot$.

Proof. C being on the bisector of the $\angle$ AOB,
CF = CG. (*Bk. I, Th. 26.*)

∴ the ⊙ described with C as centre and CF as radius will also pass through G.

Since OA is ⊥ to the radius CF at F,

∴ OA touches the ⊙ at F. (*Th.* 7, *Cor.* 2.)

Similarly, OB touches the ⊙ at G.

Thus, OA and OB are both tangents to the ⊙. ... (1)

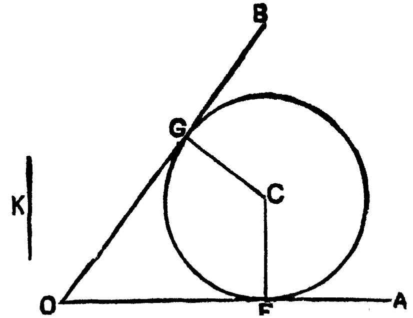
Again, since DF is a rectangle, by construction,

∴ CF = DO = K ... ... (2)

Hence, from (1) and (2), the circle described with C as centre and CF as radius is the reqd. ⊙.

Note. The following is the *analysis* of the problem :—

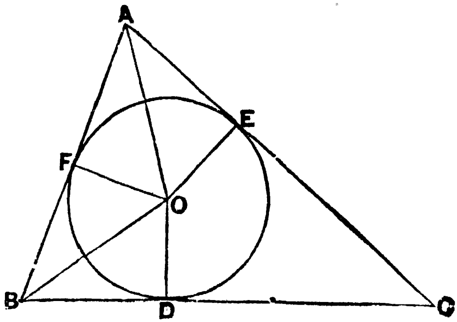
Assume a ⊙ to be described touching **OA** and **OB** at **F** and **G** respectively, and having its radius = **K**; and let **C** be the centre of the ⊙. Join **CF**, **CG**. Then **CF** and **CG**, being radii drawn to the pts. of contact, are respectively ⊥ to **OA** and **OB**. (*Th.* 7.) Now, since **CF** = **CG**,



∴ **C** lies on the bisector of the ∠**AOB**. (*Bk. I, Th.* 26) ; and also since **CF** = **K**, ∴ **C** lies on a str. line which is ∥ to **OA** and at a distance = **K** from it. Hence, if these two lines be drawn, the pt. where they intersect will be the centre of the reqd. circle.

Problem 5. (Euc. IV. 4.)

To inscribe a circle in a given triangle.



Let ABC be the given △.

It is required to describe a ⊙ which shall touch the sides of the △ ABC.

Cons. Draw the bisectors of the ∠s A and B, and let them meet at O.

Draw OD, OE, OF ⊥ to BC, CA, AB respectively.

Then the ⊙ described with O as centre and OF as radius will be the reqd. ⊙.

Proof. Since O is on the bisector of the ∠ A,

∴ OF = OE. (*Bk. 1. Th. 26.*)

Also, since O is on the bisector of the ∠ B,

∴ OF = OD.

Hence, the ⊙ described with O as centre and OF as radius will also pass through E and D.

Again, since BC is ⊥ to the radius OD at D,

∴ BC touches the ⊙ at D. (*Th. 7, Cor. 2.*)

Similarly, **CA** and **AB** touch the ⊙ at **E** and **F** respectively.

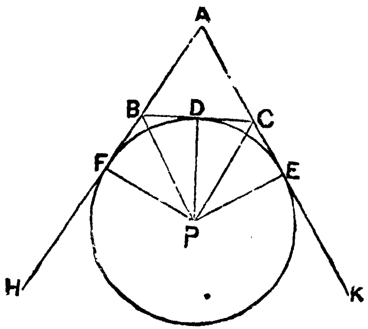
Hence the ⊙**DEF** is the reqd. ⊙.

Analysis:—*Assume* the reqd. ⊙ to be described with its centre at **O** and touching **BC, CA, AB** at **D, E, F** respectively. Join **OD, OE, OF**. Then, since **OD** is the radius drawn to the pt. of contact of the tangent **BC**, ∴ **OD** is ⊥ to **BC**, (Th. 7); similarly, **OE, OF** are ⊥ to **CA, AB** respectively. Now, since **OD = OF** ∴ **O** lies on the bisector of the ∠**B**; also, since **OE = OF**, ∴ **O** lies on the bisector of the ∠**A**. (Bk. I, Th. 26). Hence, the pt. where these two bisectors meet must be the centre of the required circle.

Note. The circle inscribed in a triangle, the centre of the circle and its radius are respectively called the **in-circle**, the **in-centre** and the **in-radius** of the triangle.

Problem 6.

To construct a circle touching one side of a given triangle and the other two sides produced.



Let ABC be the given △, and let the two sides AB, AC be produced to H, K respectively.

It is required to describe a ⊙ touching BC, BH and CK.

Cons. Draw the bisectors of the ∠s CBH and BCK, and let them meet at P.

Draw PD, PF, PE ⊥ to BC, BH and CK respectively.

Then the ⊙ described with P as centre and PD as radius will be the required ⊙.

Proof. Since P is on the bisector of the ∠CBH,

∴ PD = PF. (*Bk. I. Th. 26.*)

Also, since P is on the bisector of the ∠BCK,

∴ PD = PE.

Hence, the ⊙ described with P as centre and PD as radius will also pass through F and E.

Again, since BH is ⊥ to the radius PF at F,

∴ BH touches the ⊙ at F. (*Th. 7, Cor. 2.*)

Similarly, BC and CK touch the ⊙ at D and E respectively.

Hence the ⊙ DEF is the reqd. ⊙.

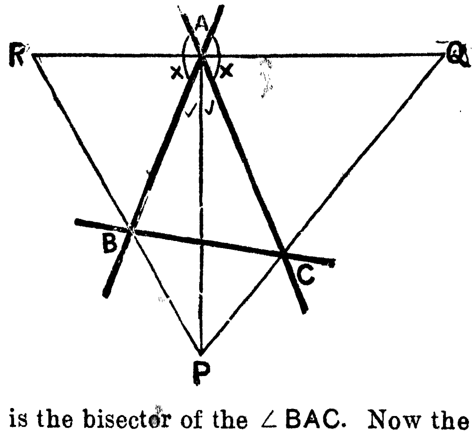
Note 1. The *Analysis* is left as an exercise for the student.

Note 2. Since PF = PE, the point P lies on the bisector of the ∠A. (*Bk. I, Th. 26.*)

Hence, *the bisector of one angle of a triangle and the bisectors of the exterior angles at the other two vertices are concurrent.*

Note 3. The circle which touches one side of a triangle and the other two sides produced is called an **escribed circle** of the triangle. The centre of an escribed circle of a triangle is called an **ex-centre** of the triangle.

Cor. If P, Q, R be the ex-centres of a △ ABC opp. to A, B, C respectively, then PA, QB, RC are respectively ⊥ to QR, RP, PQ.

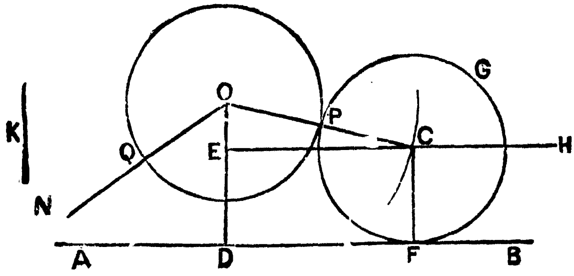


Let the bisectors of the exterior ∠s at A, B, C form the △PQR, as in the adjoining diagram. Then P, Q, R are the ex-centres of the triangle, opposite to A, B, C respectively.

Join AP; then AP is the bisector of the ∠ BAC. Now the ∠ BAR = the ∠ CAQ, (being halves of vertically opp. ∠s); also the ∠ PAB = the ∠ PAC; hence the ∠ PAR = the ∠ PAQ, and ∴ PA is ⊥ to QR. Similarly QB is ⊥ to RP, and RC is ⊥ to PQ.

Problem 7.

To construct a circle, with a given radius touching a given circle and a given straight line.



Let the ⊙ with centre O be the given ⊙, AB the given str. line, and K the given radius.

It is required to describe a circle which shall touch the circle (O) as well as the straight line AB, and have a radius = K.

Cons. Draw OD ⊥ to AB, and from DO cut off DE = K; through E draw EH ∥ to AB.

Take a radius OQ of the given ⊙ and produce it to N, making QN = K. Then ON is the sum of the radii of the given and the required ⊙s.

With O as centre and ON as radius describe an arc cutting EH at C.

Join OC cutting the given ⊙ at P, and draw CF ⊥ to AB.

Then the ⊙ described with C as centre and CP as radius will be the reqd. ⊙.

Proof. Since $OC = ON$, of which $OP = OQ$,

$\therefore \quad CP = NQ = K$;

also, since DC is a rectangle, by construction,

$\therefore \quad CF = ED = K$.

Hence $CP = CF$; and $\therefore$ the ⊙ described with C as centre and CP as radius will also pass through F.

Now, since the two ⊙s meet at P which is a pt. on the line joining the centres.

$\therefore$ the ⊙s touch each other at P. (*Th. 9, Note. 2.*)
Also, since AB is $\perp$ to the radius CF at F,

$\therefore$ AB touches the ⊙ at F. (*Th. 7, Cor. 2.*)

Thus the ⊙ PFG touches both the given ⊙ and the given str. line, and has its radius $= K$;

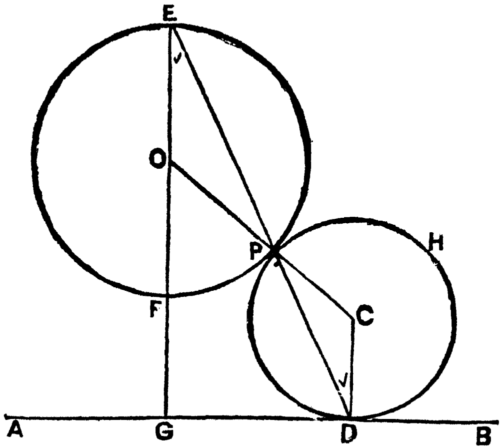
$\therefore$ this is the required ⊙. Q. E. F.

Note 1. The circle described with **O** as centre and **ON** as radius will also cut **HE** produced at some pt. With that point as centre another ⊙ may be constructed satisfying the conditions of the problem. This is left as an exercise for the student.

Note 2. If the reqd. ⊙ is ***assumed*** to be drawn, it is easy to see that the distance of its centre from **O** is equal to the sum of the radii, and the distance from **AB = K.** Hence follows the construction.

Problem 8.

To construct a circle touching a given circle and also touching a given straight line at a given point.



Let AB be the given str. line and D the given pt. in it, and let the ⊙ with centre O be the given ⊙.

It is required to describe a circle touching the circle (O) and also touching AB at D.

Cons. Draw OG ⊥ to AB cutting the ⊙ at F, and produce GO to meet the ⊙ in E.

(i) Join ED cutting the ⊙ at P; join OP.

Through D draw a perpendicular to AB, meeting OP produced in C.

Then the ⊙ described with C as centre and CP as radius will be the reqd. ⊙.

Proof. Since EG and CD are both $\perp$ to AB,

$\therefore$ they are $\parallel$,

and $\therefore$ the $\angle$ OEP = the $\angle$ CDP ;

also, the $\angle$ OPE = the $\angle$ CPD. (*Vertically opp.* $\angle$*s.*)

But the $\angle$ OEP = the $\angle$ OPE,

($\because$ OP = OE),

$\therefore$ the $\angle$ CDP = the $\angle$ CPD,

and $\therefore$ CP = CD.

Hence the $\odot$ described with C as centre and CP as radius will also pass through D.

Again, since the two $\odot$s meet at P which is on the line joining the centres, $\therefore$ the $\odot$s touch each other at P ;

(*Th.* 9, *Note.* 2.)

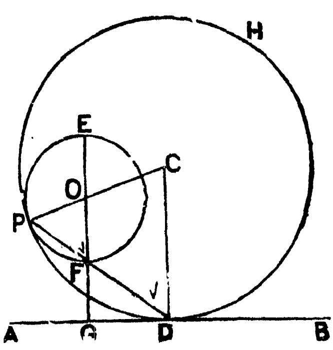
also, since AB is $\perp$ to the radius CD at D,

$\therefore$ AB touches the $\odot$ at D. (*Th.* 7, *Cor.* 2.)

Thus, the $\odot$ PDH touches the given $\odot$, and also touches the given str. line at the given pt. D ;

$\therefore$ this is the reqd. $\odot$. Q. E. F.

(ii) Join DF and produce it to meet the $\odot$ at P ; P. Join PO.



Through D draw a $\perp$ to AB, meeting PO produced in C.

Then the $\odot$ described with C as centre and CP as radius will be the required $\odot$.

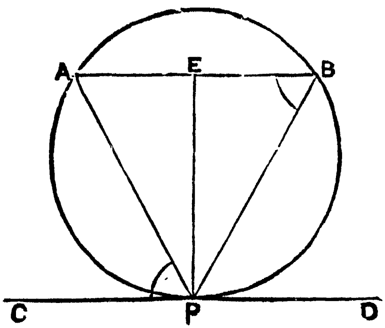
[The proof is left as an exercise for the student.]

Note 1. Let the required circle be *assumed* to be drawn, as in the first diagram; let **C** be its centre and **P** the pt. where it touches the given $\odot$. Then **OC** passes through **P**, and **CD** is $\perp$ to **AB**. Join **DP** and produce it to meet the given $\odot$ in **E**; join **EO**. Now, it is easily seen that the $\angle$**CDE** = the $\angle$**OED**; $\therefore$ **EO** is $\parallel$ **CD**, and $\therefore$ **EO** produced cuts **AB** at rt. $\angle$s. We thus find that the pt. of contact of the two $\odot$s lies on the line **DE**, where **OE** is that radius of the given $\odot$ which, when produced through **O**, cuts **AB** at rt. $\angle$s. The position of **P** being known, that of **C** is found at once, for it lies on **OP** produced and also on the $\perp$ to **AB** through **D**.

Note 2. Let the reqd. circle be *assumed* to be drawn, as in the second diagram; let **C** be its centre and **P** the pt. where it touches the given $\odot$. Then **CO** produced passes through **P**, and **CD** is $\perp$ to **AB**. Join **DP** cutting the given $\odot$ at **F**; join **OF**. Now, it is easily seen that the $\angle$**CDP** = the $\angle$**OFP**; $\therefore$ **OF** is $\parallel$ to **CD**, and $\therefore$ **OF** produced cuts **AB** at rt. $\angle$s. We thus find that the pt. of contact of the two $\odot$s lies on the line **DF** produced, where **OF** is that radius of the given $\odot$ which, when produced through **F**, cuts **AB** at rt. $\angle$s. The position of **P** being known that of **C** is found at once, for it lies on **PO** produced and also on the $\perp$ to **AB** through **D**.

Problem 9.

To construct a circle passing through two given points and touching a given straight line.



Let A, B be the two given pts. and CD the given str. line.

It is required to describe a circle passing through A and B, and also touching the line CD.

Join AB. Then AB is either ∥ to CD ; or meets CD when produced.

(i) Let AB be ∥ to CD.

Cons. Draw the ⊥ bisector of AB meeting CD in P.

Then the ⊙ passing through the pts. A, B, P will be the reqd. ⊙.

Proof. Join PA, PB.

Now, P being on the ⊥ bisector of AB, PA = PB,

and ∴ the ∠PBA = the ∠PAB.

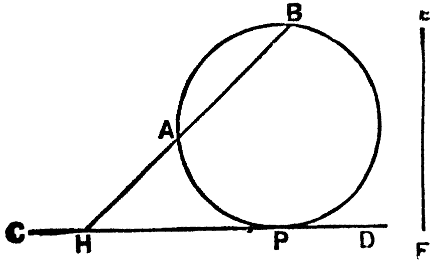
Since AB is ∥ to CD, ∴ the ∠CPA = the ∠PAB.

Hence, the $\angle$ CPA = the $\angle$ PBA, in the alt. segment;

$\therefore$ CP touches the $\odot$ at P. (*Th. 16, Cor.*)

Hence ABP is the reqd. $\odot$. Q. E. F.

(ii) Let BA produced meet CD in H.



Cons. Find a str. line EF the square on which is equal in area to the rect. BH. HA, (*Bk. II. Prob. 3.*)

From HD cut off HP = EF.

Then the $\odot$ passing through the pts. A, B, P will be the reqd. $\odot$.

Proof. Since HP^2 = HA. HB,

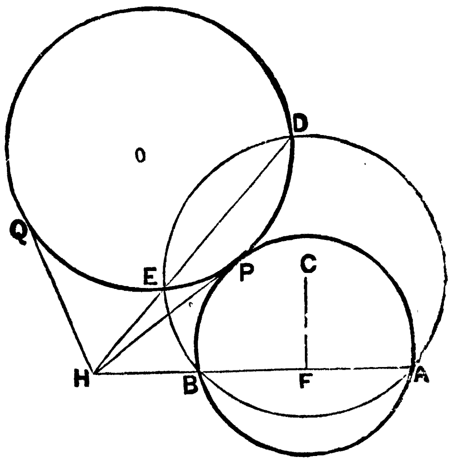
$\therefore$ HP touches the $\odot$ at P. (*Th. 17, Cor. 2.*)

Hence ABP is the reqd. $\odot$. Q. E. F.

Note. HP may as well be cut off from HC produced. Thus there are two $\odot$s satisfying the given conditions.

Problem. 10.

To construct a circle passing through two given points and touching a given circle.



Let A, B be the two given pts. and let the ⊙ with centre O be the given ⊙.

It is required to describe a circle passing through A and B, and also touching the circle (O).

Join AB and draw FC, the ⊥ bisector of AB.

Then, FC may pass through O or may not.

(i) Let FC not pass through O.

Cons. With any pt. C on FC as centre, and CA or CB as radius, describe a ⊙ cutting the given ⊙ in D and E. Join DE and produce it to meet AB produced in H.

Draw HP to touch the given ⊙ at P. (*Th. 13, Note.*)

Construct a ⊙ through the pts. A, B, P ; then this will be the reqd. ⊙.

Proof. Since the chords AB, DE of the ⊙ ABD intersect in H,

$\therefore$ HA.HB = HD.HE. (*Th.* 17.)

Again, since HP touches the given ⊙ at P and HD cuts it in D and E,

$\therefore$ HP^2 = HD.HE. (*Th.* 17, *Cor.* 1.)

Hence, HA.HB = HP^2,

and $\therefore$ HP touches the ⊙ ABP at P. (*Th.* 17, *Cor.* 2.)

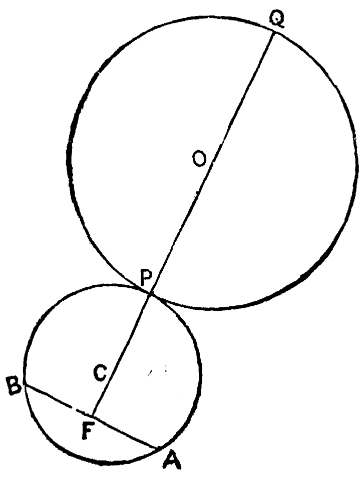
Now, the ⊙ ABP and the given ⊙ have a common tangent at P,

$\therefore$ the two ⊙s touch each other at P. (*Th.* 9, *Note* 3.

Hence the ⊙ ABP is the reqd. ⊙. Q. E. F.)

[Similarly, if the other tangent HQ to the given ⊙ be drawn, it may be proved as above that the ⊙ through A, B, Q is another ⊙ satisfying the given conditions.]

(ii) Let FC pass through O.



Cons. Let FO cut the given ⊙ at P.

Describe a ⊙ passing through the pts. A, B, P ; then this will be the reqd. ⊙.

Proof. Since FP is the ⊥ bisector of AB,

∴ the centre of the ⊙ ABP lies in FP.

Hence, P is a pt. on the line joining the centres of the two ⊙s ;

and since the ⊙s meet at P ;

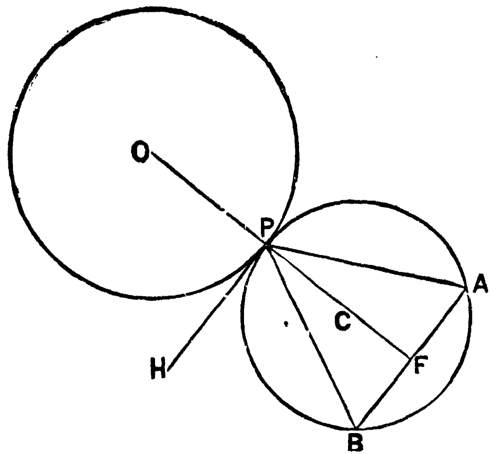
∴ they touch each other at P. (*Th. 9, Note 2.*)

Hence the ⊙ ABP is the reqd. ⊙. Q. E. F.

[If FO be produced to meet the given ⊙ in Q, it may be proved as above, that the ⊙ through ABQ is another ⊙ satisfying the given conditions.]

Analysis :—*Assume* the required ⊙ to be drawn, touching the given ⊙ at **P**. Then the two ⊙s have a common tangent at **P** ; let this tangent meet **AB** produced in **H**. ∴ **HP² = HA. HB**, (Th. 17, Cor. 1). Through **H** let any secant be drawn cutting the given ⊙ in **E** and **D** ; then **HD. HE = HP²** (Th. 17, Cor. 1). Hence **HA. HB = HD. HE**, which shows that the pts. **A**, **B**, **E**, **D** are concyclic (Th. 17, Cor. 3). Hence, if a ⊙ be constructed passing through **A**, **B** and also cutting the given ⊙ in two pts. **D**, **E**, then the pt. in which **DE** intersects **AB** is also the pt. in which the common tangent to the given and the reqd. ⊙s intersects **AB** ; whence the construction follows.

If the common tangent at **P** *be* ‖ *to* **AB**, *it will not meet* **AB** *produced.*

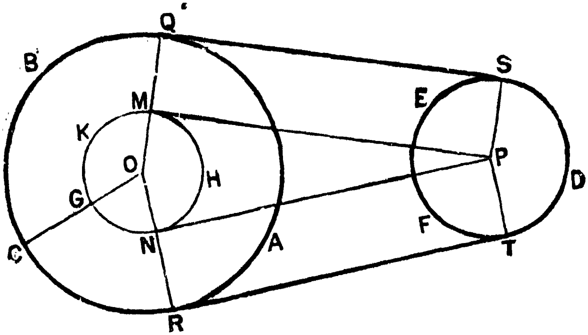


Join **PA**, **PB.** Since **PH** is ‖ to **AB**, ∴ the ∠**HPB** = the ∠**PBA**; also the ∠**HPB** = the ∠**PAB**, in the alt. segment.

Hence the ∠ **PBA** = the ∠**PAB**, and ∴ **PA** = **PB**. Hence **P** is on the ⊥ bisector of **AB**. The centre **C** of the ⊙**APB** also lies on the ⊥ bisector of **AB**. Hence both **C** and **P** lie on the ⊥ bisector of **AB**, and ∴ **PC** produced bisects **AB** at rt. ∠s. But **CP** produced also passes through **O**. Hence **OP** produced bisects **AB** at rt. ∠s. In this case therefore the ⊥ bisector of **AB** passes through **O** and cuts the given ⊙ in the pt. at which the reqd. ⊙ touches it; whence follows the construction.

Problem 11.

To draw a common tangent to two given circles.



Let ABC and DEF be two given ⊙s whose centres are O and P respectively, the radius of the first ⊙ being greater than the radius of the second.

It is required to draw a common tangent to the ⊙s ABC and DEF.

Cons. Draw any radius OC of the ⊙ ABC.

(i) From CO cut off CG = the radius of the ⊙ DEF.

With O as centre and OG as radius describe the ⊙ GKH.

Draw PM, PN tangents to the ⊙ GKH. (*Th. 13, Note.*)

Join OM, ON and produce them to meet the ⊙ ABC at Q, R respectively.

In the ⊙ DEF, draw the radius PS ∥ to OQ, and the radius PT ∥ to OR.

Join QS, RT.

Then QS as well as RT are common tangents to the ⊙s ABC and DEF.

Proof. Since PM touches the ⊙ GKH at M,

∴ the ∠OMP is a rt. ∠ ; (*Th. 7.*)

and ∴ the ∠PMQ is also a rt. ∠.

Now, OQ = OC, and of these the parts OM, OG are equal ; ∴ MQ = OG.

Hence PS = MQ ; and PS is also ∥ to MQ, by construction, ∴ PSQM is a parallelogram. (*Bk. I, Th. 22.*)

Also, since the ∠PMQ of this parallelogram is a rt. ∠.

∴ all its ∠s are rt. ∠s.

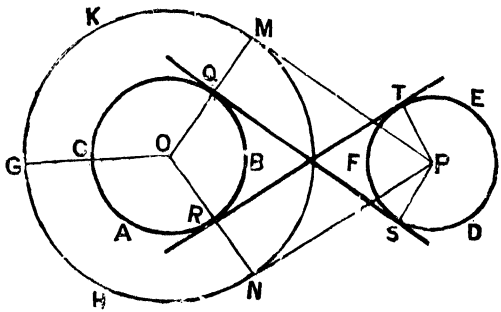
Thus, the ∠s OQS and PSQ are rt. ∠s ;

∴ QS is a tangent to the ⊙ ABC as well as to the ⊙DEF. (*Th. 7, Cor. 2.*)

In the same way it may be proved that RT is a tangent to both the given ⊙s. Q. E. F.

(ii) From OC produced cut off CG = the radius of the ⊙ DEF.

With O as centre and OG as radius describe the ⊙GKH,



Draw PM, PN tangents to the ⊙ GKH. (*Th. 13, Note.*)

Join OM, ON cutting the ⊙ABC at Q and R respectively.

In the ⊙DEF, draw the radius PS ∥ to MO, and the radius PT ∥ to NO.

Join QS, RT.

Then QS as well as RT are common tangents to the ⊙s ABC and DEF.

[The proof is left as an exercise for the student.]

Note 1. It is clear that there are *four* common tangents to two given ⊙s, if the ⊙s do not cut each other. A common tangent whose pts. of contact are on the same side of the line joining the centres is called a **direct common tangent**; whilst a common tangent whose pts. of contact are on the opposite sides of the line joining the centres is called a **transverse common tangent.** We have *direct common tangents* in the first diagram, and *transverse common tangents* in the second.

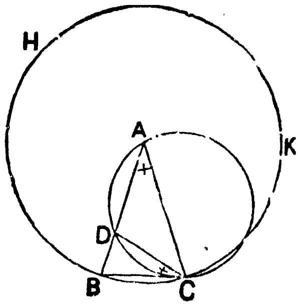
Note 2. If the direct common tangents be produced to meet at **V**, then each of the pts. **P** and **O** is equally distant from the lines **QV** and **RV**, and ∴ both of them lie on the bisector of the ∠**QVR**.

Hence **OP** produced passes through **V**. Thus, *the direct common tangents to two given ⊙s and the line joining their centres are concurrent.*

Note 3. In the second diagram, each of the pts. **P** and **O** is equally distant from **TR** and **QS**, and ∴, if **TR** and **QS** intersect at **W**, both **P** and **O** lie on the bisector of the ∠s **TWS** and **QWR**. Hence **OP** passes through **W**. Thus, *the transverse common tangents to two given ⊙s intersect on the line joining their centres.*

Problem 12. (Euc. IV. 10.)

To construct an isosceles triangle having each of the angles at the base double of the vertical angle.



Cons. Take any str. line AB and divide it at D so that AB. BD = AD^2. (*Bk. II. Sec. IV. Prop. 7.*)

With A as centre and AB as radius describe the ⊙HBK.

In this ⊙ place the chord BC = AD.

Join AC.

Then ABC is the △ reqd.

Proof. Join CD, and circumscribe a ⊙ about the △ADC.

Since $BC^2 = AD^2$, ∴ BC^2 = AB. BD.

∴ BC touches the ⊙ ADC at C. (*Th. 17, Cor. 2.*)

Hence the ∠ BCD = the ∠ DAC, in the alt. segment.

Hence, the ∠ BDC = the ∠ BCA,

because each of them = the sum of the ∠s DAC, ACD.

But the $\angle$ ABC = the $\angle$ BCA ;

($\because$ AC = AB),

$\therefore$ the $\angle$ ABC = the $\angle$ BDC.

Hence, DC = BC = DA ;

and $\therefore$ the $\angle$ DAC = the $\angle$ DCA.

Hence, DC is the bisector of the $\angle$ BCA,

and $\therefore$ the $\angle$ BCA = twice the $\angle$ BCD

= twice the $\angle$ BAC.

Hence the $\angle$ CBA also = twice the $\angle$ BAC.

Thus, ABC is the required $\triangle$. Q. E. F.

Analysis :—*Assume* **ABC** to be a $\triangle$ such that each of its $\angle$s **B** and **C** is double the $\angle$ **A**.

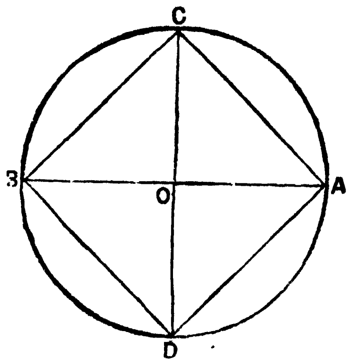
Let **CD** bisect the $\angle$**C**. Then the $\angle$**BCD** = $\frac{1}{2}$ the $\angle$**BCA** = the $\angle$**BAC**. Hence **BC** must touch the circum-$\odot$ of the $\triangle$**ADC** at **C**. (*Th.* 16, *Cor.*)

$\therefore$ **BC^2 = AB.BD.** The $\angle$**DAC** = the $\angle$**DCA**, (because each of them is $\frac{1}{2}$ the $\angle$**BCA**), $\therefore$ **DC = DA** ; also, since the $\angle$**BDC** = the sum of the $\angle$s **DAC**, **DCA** = twice the $\angle$**DAC** = the $\angle$**ABC**, $\therefore$ **BC = DC.** Hence **BC = DA**, and $\therefore$ **AB. BD = AD^2** ; which determines the position of **D.** Then the position of **C** also is determined, because the distance of **C** from **A = AB**, and its distance from **B = DA.**

Note. Evidently 5 times the $\angle$**A** = two rt. $\angle$s. Hence the $\angle$**A** = one-fifth of two rt. $\angle$s, and $\therefore$ each of the $\angle$s **ABC**, **ACB** = two-fifths of two rt. $\angle$s = *one-fifth of four rt.* $\angle$*s.*

Problem 13. (Euc. IV. 6 and 7.)

To construct a regular figure of four sides in or about a given circle.



Let the ⊙ with centre O be the given ⊙.

It is required to construct an equilateral and equiangular quadrilateral in or about the ⊙ (O).

Cons. Take any diameter AB, and draw the diameter CD ⊥ to AB.

(i) Join AC, CB, BD, DA.

Then ABCD is a regular quadl. inscribed in the given ⊙.

Proof. The four ∠s at O are equal, each of them being a rt. ∠;

∴ the four arcs AC, CB, BD, DA are equal.

(*Th. 4, Cor. 1.*)

Hence the chords AC, CB, BD, DA are equal;

(*Th. 5, Cor. 1.*)

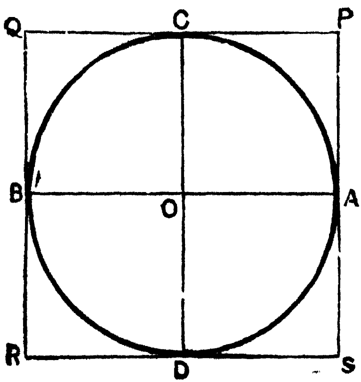
i.e., the quadrilateral ACBD is equilateral.

Again, the ∠ ACB being in a semi-⊙, is a rt. ∠ ; and similarly, each of the ∠s at B, D, A is a rt. ∠.

∴ all the ∠s of the quadl. are equal Thus the quadl. ACBD is both equilateral and equiangular, and it is also *inscribed* in the given ⊙.

(ii) Draw tangents to the ⊙ at A, C, B, D forming the quadl. PQRS.

Then PQRS is a regular quadl. circumscribed about the given ⊙.



Proof. The ∠s at A, C, B, D are all rt. ∠s. (*Th. 7.*)

Hence the ∠OAP = the ∠DOA;

∴ PS is ∥ to CD.
Similarly, QR is ∥ to CD.

Hence PS, CD, QR are ∥ to one another;

and similarly QP, BA, RS are ∥ to one another.

Now, since BP is a par^{m}.

∴ PQ = AB ;
and similarly, QR = CD.

∴ PQ = QR.

Similarly, QR = RS, and RS = SP.

Again, since CA is a par^{m}.,

$\therefore$ the $\angle$ APC = the $\angle$ AOC = a rt. $\angle$;

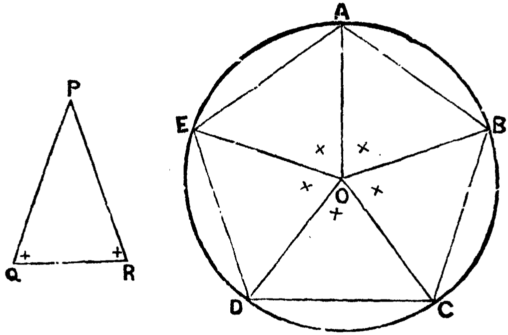
similarly, each of the $\angle$s at Q, R, S is a rt. $\angle$.

Hence the quadl. PQRS is also equiangular. Thus the quadl. PQRS is both equilateral and equiangular. and it is also *circumscribed* about the given $\odot$. Q. E. F.

Note. It is easy to see that a regular quadrilateral is a square. In the diagrams of Problem 13, each of the quadls. **ACBD** and **PQRS** is a rectangle with adjacent sides equal, and is $\therefore$ a square.

Problem 14. (Euc. IV. 11 and 12.)

To construct a regular figure of five sides in or about a given circle.



Let the ⊙ with centre O be the given ⊙.

It is required to construct an equilateral and equiangular pentagon in or about the circle (O).

Cons. Construct an isosceles △ PQR having each of the ∠s at Q and R double of the ∠ at P. (*Prob. 12.*)

Draw any radius OA, and then draw the radius OB making the ∠ AOB = the ∠ Q.

Now, successively draw the radii OC, OD, OE making the ∠s BOC, COD, DOE each = the ∠ Q.

(i) Join AB, BC, CD, DE, EA. Then ABCDE is a regular pentagon inscribed in the given ⊙.

Proof. Each of the ∠s at Q, R is one-fifth of four rt. ∠s. (*Prob. 12, Note.*)

Hence each of the ∠s AOB, BOC, COD, DOE is one-fifth of four rt. ∠s;

$\therefore$ the $\angle$EOA, which together with those four $\angle$s makes up the four rt. $\angle$s at O, must also be = one-fifth of four rt. $\angle$s.

Thus the five $\angle$s at O are equal;

$\therefore$ the five arcs AB, BC, CD, DE, EA are equal. (*Th. 4, Cor. 1.*)

Hence the five chords AB, BC, CD, DE, EA are equal. (*Th. 5, Cor. 1.*)

The pentagon ABCDE is $\therefore$ equilateral.

Again, the arc AEDC = the arc BAED;

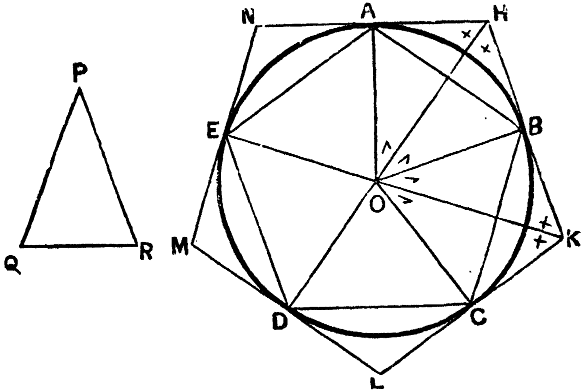
($\because$ each of them = 3 times the arc AB),

$\therefore$ the $\angle$ABC = the $\angle$BCD. (*Th. 10, Cor. 2.*)

Similarly, every two consecutive $\angle$s of the pentagon are equal.

Hence the pentagon ABCDE is both equilateral and equiangular, and it is also *inscribed* in the given $\odot$.

(ii) Draw tangents to the $\odot$ at A, B, C, D, E forming the pentagon HKLMN. Then HKLMN is a regular pentagon circumscribed about the $\odot$.



Proof. Join OH, OK.

It may be proved, as before, that the five radii OA, OB, OC, OD, OE divide the four rt. $\angle$s at O into five equal parts.

Also the $\angle$s at A, B, C, D, E are rt. $\angle$s. (*Th.* 7.)

Now, since the tangents AH, BH meet at H,

$$\left.\begin{array}{lrcl} & \therefore \quad AH &=& BH, \\ & \text{the } \angle AOH &=& \text{the } \angle BOH, \\ \text{and} & \text{the } \angle OHA &=& \text{the } \angle OHB. \end{array}\right\} \quad (Th.\ 8.)$$

$$\left.\begin{array}{lrcl} \text{Hence,} & \text{the } \angle BOH &=& \tfrac{1}{2} \text{ the } \angle AOB, \\ \text{and} & \text{the } \angle OHB &=& \tfrac{1}{2} \text{ the } \angle AHB. \end{array}\right\}$$

$$\left.\begin{array}{lrcl} \text{Similarly,} & \text{the } \angle BOK &=& \tfrac{1}{2} \text{ the } \angle BOC, \\ \text{and} & \text{the } \angle OKB &=& \tfrac{1}{2} \text{ the } \angle BKC. \end{array}\right\}$$

In the Δs OBH, OBK, we have

(1) the $\angle$OBH = the $\angle$OBK, (*rt.* $\angle$*s.*)

(2) the $\angle$BOH = the $\angle$BOK. (halves of the equal $\angle$s AOB, BOC)

and (3) the side OB common ;

$\therefore$ the two Δs are congruent.

Hence BH = BK ... ... (α)

and the $\angle$OHB = the $\angle$OKB ... ... (β)

From (α), HK = 2BH; and in the same way it may be proved that NH = 2AH.

$\therefore$ NH = HK, ($\because$ AH = BH);

similarly HK = KL, KL = LM, LM = MN, and MN = NH.

Hence the pentagon HKLMN is equilateral.

From (β), halves of the $\angle$s AHB and BKC are equal,

$\therefore$ the $\angle$AHB = the $\angle$BKC;

similarly, every two consecutive $\angle$s of the pentagon are equal.

Hence the pentagon **HKLMN** is also equiangular.

Thus the pentagon **HKLMN** is both equilateral and equiangular, and it is also *circumscribed* about the given $\odot$.

Q. E. F.

Note. An angle of an equilateral $\triangle$ is one-third of two rt. $\angle$s, and $\therefore$ one-sixth of four rt. $\angle$s. Hence, the four rt. $\angle$s at the centre of a $\odot$ may be divided into six equal parts, in the manner exemplified in the present problem; which enables us to construct a regular figure of six sides in or about the cirele. This is left as an exercise for the student.

EXERCISE (26)

1. Construct a circle passing through a given point and touching a given straight line at a given point.

2. Construct a ⊙, with a given radius, passing through two given pts.

3. Construct a circle, with a given radius, which shall touch a given straight line and pass through a given point.

4. Construct a circle, with a given radius, touching two given circles.

5. Two parallel straight lines are cut by a third straight line. Construct a circle touching each of these three straight lines.

6. Find the magnitude of the angle subtended at the centre of a given circle by each of the sides of an inscribed equilateral triangle. Hence shew how to construct an equilateral triangle in a given circle.

7. Prove that tangents drawn to a circle at the vertices of an inscribed equilateral triangle will form another equilateral triangle. Hence, given a circle, shew how to construct an equilateral triangle about it.

8. Divide the circumference of a given circle into six equal parts. Prove that the chord joining any two consecutive points of division is equal to the radius of the circle

9. Construct a regular figure of six sides in a given circle.

10. Construct a regular figure of six sides about a given circle.

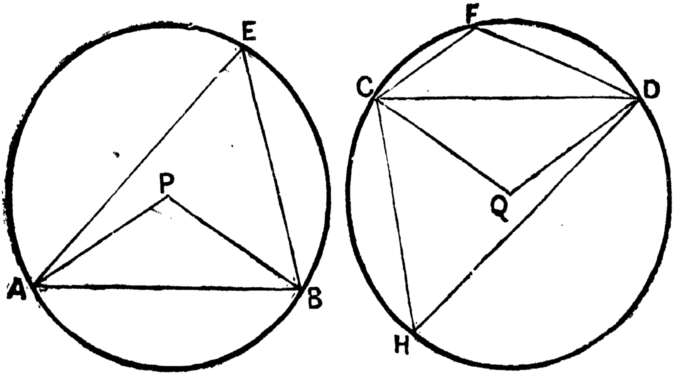
11. Construct an angle of 120°, and also an angle of 72°, and hence construct an angle of 24°.

12. Construct a regular figure of fifteen sides in a given cirlce.

SECTION IV.

MISCELLANEOUS PROPOSITIONS

1. *If two triangles standing on equal bases have vertical angles that are either equal or supplementary then their circum-circles are equal.*



Let the $\triangle$s AEB, CFD standing on equal bases AB, CD have their vertical $\angle$s AEB, CFD *supplementary*.

Let P, Q be the centres of the $\odot$s circumscribed about the $\triangle$s AEB, CFD respectively.

To prove that these two circles have equal radii.

Proof. Join PA, PB, QC, QD,

Take any pt. H on the arc conjugate to the arc CFD. Join CH, DH.

Then the $\angle$CHD = the $\angle$ AEB,

each of them being supplementary to the $\angle$CFD ;

$\therefore$ the $\angle$ CQD = the $\angle$ APB, (doubles of equal $\angle$s).

Now, since the $\angle$ PAB = the $\angle$ PBA, each of them = $\frac{1}{2}$ their sum = $\frac{1}{2}$ the supplement of the $\angle$ APB ;

similarly, each of the $\angle$s QCD, QDC

= $\frac{1}{2}$ the supplement of the $\angle$ CQD.

Hence, each of the $\angle$s PAB, PBA = each of the $\angle$s QCD, QDC.

Then the two isos. Δs PAB, QCD have their bases equal, and also the $\angle$s at the base of the one = the $\angle$s at the base of the other ;

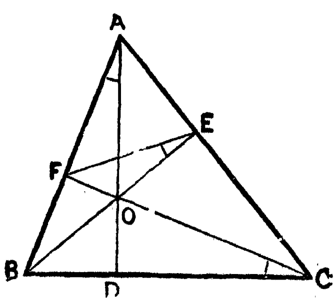
$\therefore$ these two Δs are congruent. (*Bk. I, Th. 18.*)

Hence, each of the sides PA, PB = each of the sides QC, QD.

Thus, the radii of the two $\odot$s are equal; which proves the proposition.

Note. It is obvious, from the foregoing demonstration, how to deal with the case when the two triangles have their vertical angles *equal*.

2. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



Let BE, CF be $\perp$s to the sides CA, AB respectively of the $\triangle$ABC ; and let them intersect at O.

Join AO and produce it to meet BC at D.

To prove that AD is $\perp$ to BC.

Proof. Join FE.

The $\angle$s OFA, OEA are supplementary, each of them being a rt. $\angle$;

$\therefore$ the quadl. AFOE is cyclic.

$\therefore$ the $\angle$ FAO = the $\angle$EFO, in the same segment. (α)

Again, the $\angle$BFC = the $\angle$BEC, each being a rt. $\angle$;

$\therefore$ the four pts. B, F, E, C are concyclic, (*Th. 12.*)

$\therefore$ the $\angle$FEB = the $\angle$FCB, in the same segment. (β)

Hence, from (α) and (β),

the $\angle$ BAD = the $\angle$ BCF.

Now in the $\triangle$s BAD, BCF the $\angle$ at B is common, and the $\angle$BAD = the $\angle$ BCF ;

$\therefore$ the remaining $\angle$ ADB = the remaining $\angle$ CFB = a rt. $\angle$.

Hence, AD is $\perp$ to BC. Q. E. D.

Note 1. For an alternative proof, see Book I., Sec. VIII, Prop. 3.

Note 2. The point of intersection of the perpendiculars from the vertices of a triangle to the opposite sides is called the **ortho-centre** of the triangle. Thus, in the preceding diagram, the point **O** is the ***ortho-centre*** of the triangle **ABC**.

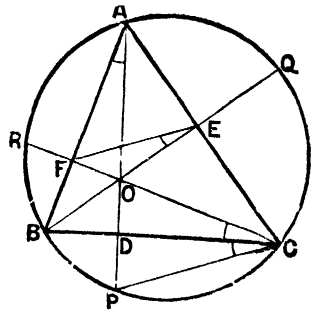
Note 3. The triangle formed by joining the feet of the perpendiculars drawn from the vertices of any given triangle to the opposite sides is called the **pedal** triangle of the given triangle. In the preceding diagram, the triangle **DEF** is the ***pedal triangle*** of the triangle **ABC**.

Cor. 1. *If* O *is the ortho-centre of a triangle* ABC, *the circum-circles of the four triangles* ABC, BOC, COA, AOB *are equal to one another.*

In the preceding diagram, the quadl. AFOE being cyclic, the ∠ FAE is supplementary to the ∠ FOE and is ∴ supplementary to the ∠ BOC. Thus, the △s BAC, BOC stand on the same base and have their vertical ∠s supplementary;

∴ the circum-⊙ of the △BOC is = that of the △ ABC (*Prop 1.*). Similarly, the circum-⊙ of each of the △s COA, AOB is = that of the △ABC Hence the four circum-⊙s are equal to one another.

Cor. 2. *If the perpendiculars* AD, BE, *from the virtices* A, B, C *of a triangle to the opposite sides mee at* O *and if* AD' BE, CF *be produced to meet the circum-circle of the triangle* ABC *in* P, Q, R *respectively, then* D, E, F *are the mid-points of* OP, OQ, OR *respectively.*



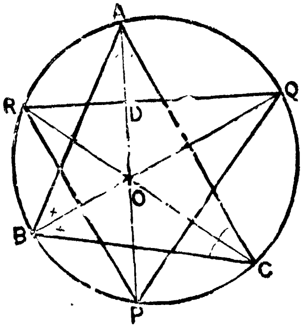
Join PC. Then the ∠ BCP = the ∠ BAP = the ∠ BCF.

Now the △s ODC, PDC are evidently congruent.

(*Bk. I, Th. 18.*)

∴ OD = DP. Similarly, OE = EQ and OF = FR.

3. *If* O *is the in-centre of a triangle* ABC, *and if* AO, BO, CO *be produced to meet the circum-circle in* P, Q, R *respectively, then* O *is the ortho-centre of the triangle* PQR.



Proof. Let AP intersect RQ at D.

AB, BC, CA are tangents to the in-⊙ of the △ABC ;

∴ AO, BO, CO are the bisectors of the ∠s A, B, C. (*Th. 8.*)

Now, the ∠PRC = the ∠PAC (*Th. 11.*)
$= \frac{1}{2}$ the ∠A ;

the ∠CRQ = the ∠CBQ = $\frac{1}{2}$ the ∠B ;

and the ∠RPA = the ∠RCA = $\frac{1}{2}$ the ∠C.

Hence, in the △PRD, we have

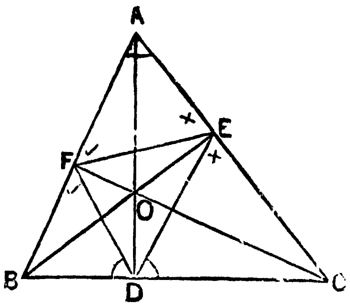
the ∠PRD + the ∠RPD = $\frac{1}{2}$ the ∠A + $\frac{1}{2}$ the ∠B + $\frac{1}{2}$ the ∠C = one rt. ∠ ;

∴ the remaining ∠PDR = one rt. ∠, and ∴ PA is ⊥ to QR.

Similarly QB is ⊥ RP, and RC is ⊥ to PQ.

Hence, O is the ortho centre of the △PQR. Q. E. D

4. *In an acute-angled taiangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle.*



Let O be the ortho-centre, and DEF the pedal △, of the acute-angled △ABC.

To prove that AD, BE, CF **bisect the** ∠s D, E, F **respectively of the** △DEF.

Proof. The ∠ADB = the ∠AEB, each b-ing a rt. ∠.

∴ the quadl. AEDB is cyclic (*Th. 12.*)

Hence, the ∠EDC = the ∠BAE; (*Th. 14, Cor. 1.*)

and similarly, the ∠FDB = the ∠CAF.

∴ the ∠EDC = the ∠FDB;

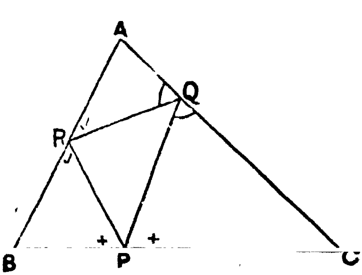
a d the whole ∠ADC = the whole ∠ADB.

∴ the remaining ∠ADE = the remaining ∠ADF.

Thus, AD bisects the ∠EDF.

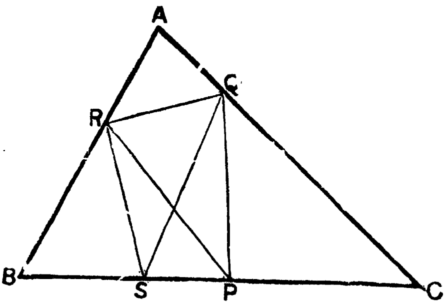
Similarly, BE and CF bisect the ∠s DEF and DFE respectively. Q. E. D.

Note. The pedal triangle of an acute-angled triangle is evidently one inscribed in that triangle ; and it is also clear, from the preceding diagram that the pedal triangle possesses the property of having every two of its sides equally inclined to the side of the given triangle on which they meet. That, of all triangles inscribed in a given triangle, it is the pedal triangle alone that possesses this property, can be proved as follows :—



Let the △ **PQR** inscribed in the acute-angled △**ABC** be such that its sides **RP**, **PQ** are equally inclined to **BC**, the sides **PQ**, **QR** are equally inclined to **CA**, and the sides **Q RP** are equally inclined to **AB**, as shewn in the accompanying diagram. Now, the six ∠s marked at **P**, **Q**, **R**, together with the ∠s at **A**, **B**, **C** make up *six* rt. ∠s ; hence, those *six* ∠s are together = 4 rt. ∠s. Hence the *three* ∠s **AQR**, **ARQ** and **QPC** (one being taken from each pair) are together = 2 rt. ∠s ; and ∴ the ∠**QPC** = the ∠**A**, which shews that the quadl. **ABPQ** is cyclic. Similarly, each of the quadls. **BCQR** and **C ARP** is cyclic. Hence, if **AP**, **BQ** and **CR** be joined, it is easy to see that the ∠**APQ** = the ∠**ABQ** = the ∠**RCQ** = the ∠**APR** ; whence the ∠**APC** = the ∠**APB** ; and ∴ **AP** is ⊥ to **BC**. Similarly **BQ** is ⊥ to **CA** and **CR** is ⊥ to **AB**. Hence the △**PQR** is no other than the pedal triangle. Thus, of all the △s that can be inscribed in the △**ABC**, it is only the pedal △ that has the above property.

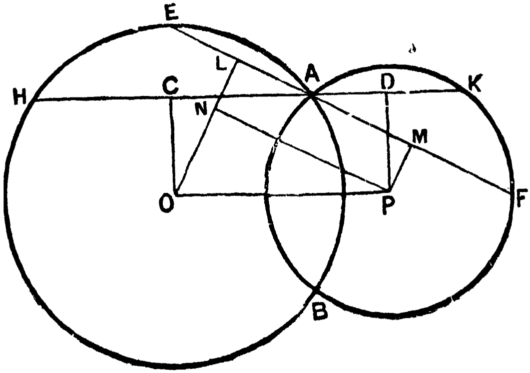
Cor. *Of all triangles that can be inscribed in an acute-angled triangle, the pedal triangle is that which has the least perimeter.*



Let ABC be an acute-angled △ ; then, of all the △s that can be inscribed in it, *either* the pedal △, *or* some one of the *others*, must have the least perimeter. Let PQR be *any* of the inscribed △s *other than* the pedal △, as in the above diagram. Since PQR is not the pedal △, it must have *at least* one pair of sides that are *not* equally inclined to that side of the △ABC on which they meet ; let RP, PQ be that pair. Suppose then that S is the pt. in BC such that RS, SQ are equally inclined to it ; then RS + SQ is < RP + PQ, (*Bk. I, Sec. VIII, Prop. 8.*) and ∴ the perimeter of the △RSQ is < that of the △RPQ. The △ PQR is ∴ *not* one having the least perimeter. Hence, the △ of minimum perimeter is clearly *no other* than the pedal △ ; or, which is the same thing, it is the pedal △ that has the minimum perimeter.

5. *Two circles whose centres are* O *and* P *cut each other at the points* A *and* B *; then, of all the lines drawn*

through A *or* B *and terminated by the two circumferences, the greatest is that which is parallel to* OP.



Let HK be the line through A ∥ to OP, and let EF be *any* other line through A, as in the above diagram.

To prove that HK is > EF.

Proof. Draw OC, PD ⊥ to HK; OL, PM ⊥ to EF, and PN ⊥ to OL.

Since OC is ⊥ to HA.

∴ HA = 2AC. (*Th. 1.*)

Similarly, AK = 2AD;

$$\left.\begin{array}{ll} \therefore & HK = 2CD. \\ \text{In the same way,} & EF = 2LM. \end{array}\right\}$$

Now CP and NM are evidently rectangles;

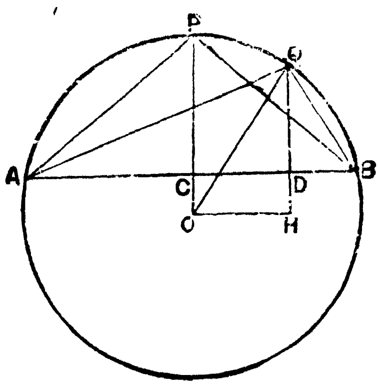
$$\left.\begin{array}{ll} \therefore & CD = OP, \\ \text{and} & LM = NP. \end{array}\right\}$$

But, in the rt. ∠d △ OPN,

the hypotenuse OP > the side NP;

∴ CD > LM, and ∴ HK > EF. Q. E. D.

6. *Of all triangles standing on the same base and having the same vertical angle, the isosceles is that which has the greatest area and the greatest perimeter.*



Let the isos. △APB stand on the base AB; and let AQB be *any other* △ on the *same* base, having the vertical ∠AQB = the ∠APB.

(i) **To prove that the area of the △APB is greater than the area of the △AQB.**

Proof. Since the ∠APB = the ∠AQB,

∴ the four pts. A, P, Q, B are concylic. (*Th. 12.*)

Let O be the centre of the ⊙ passing through these four points, and draw OC, QD ⊥ to AB.

Then OC is the ⊥ bisector of AB; (*Th. 1.*)

and since PA = PB, ∴ OC produced must pass through P.

Draw OH ⊥ to QD produced; join OQ.

Now, in the rt. ∠d △OQH, the hypotenuse OQ > QH.

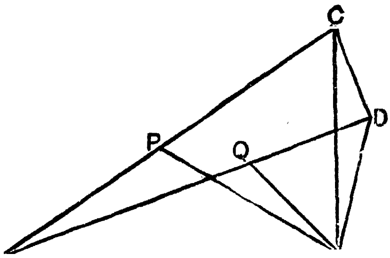
Hence OP > QH
and, CH being a rectangle, CO = DH ;

∴ PC > QD.

Now, the area of the △APB = ½ the rect. AB. PC ;
and that of the △AQB = ½ the rect. AB. QD.

Hence the area of the △APB > the area of the △AQB.

(ii) **To prove that the perimeter of the △APB is greater than the perimeter of the △AQB.**



Proof. Produce AP to C making PC = PB, and AQ to D making QD = QB.

Join CB, DB, CD.

Since AP = PB = PC, ∴ the ⊙ described on AC as diameter will pass through B ;

∴ the ∠ABC is a rt. ∠.

Now, since PC = PB, ∴ the ∠PBC = the ∠PCB,
and ∴ the ∠APB = double the ∠ACB ;
similarly, the ∠AQB = double the ∠ADB.
Hence, the ∠ACB = the ∠ADB ;
and ∴ the quadl. ACDB is cyclic. (*Th.* 12.)

Hence, the $\angle$ ADC = the $\angle$ ABC (*Th. 11.*)

= a rt. $\angle$.

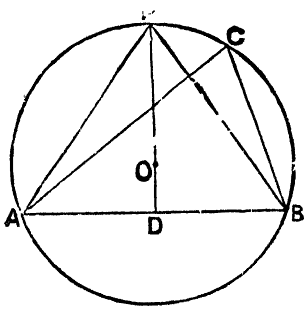
Now, in the rt. $\angle$d $\triangle$ADC,

the hypotenuse AC > the side AD;

$\therefore$ AP + PB > AQ + QB.

Hence the perimeter of the $\triangle$APB is > the perimeter of the $\triangle$AQB. Q. E. D.

7. *Of all triangles that can be inscribed in a given circle, that which has the greatest area or the greatest perimeter is equilateral.*



Let ABC be the given $\odot$ of which the centre is O.

(i) **To prove that of all the $\triangle$s that can be inscribed in the $\odot$ ABC, that which has the greatest area is equilateral.**

Proof. The $\triangle$ which has the greatest area must be *either* equilateral *or* non-equilateral.

Let ABC be *any* non-equilateral $\triangle$ inscribed in the given $\odot$.

It must then have *at least* one pair of unequal sides; let AC, CB be that pair.

Draw OD ⊥ to AB, and produce DO to meet the ⊙ in P; join PA, PB.

Now, PD is the ⊥ bisector of AB (*Th. 1.*)

∴ PA = PB.

Hence, of the two △s APB, ACB which stand on the same base AB and have equal vertical ∠s, the △APB is isosceles.

Hence the area of the △APB is > that of the △ACB (*Last Prop.*)

the △ACB ∴ has *not* the maximum area.

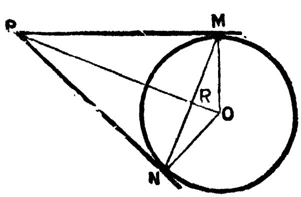
Hence, the △ of maximum area inscribed in the ⊙ can *not* be non-equilateral;

∴ it must be equilateral.

(ii) **To prove that of all the △s that can be inscribed in the ⊙ ABC, that which has the greatest perimeter is equilateral.**

[The Proof is similar to the above and is left as an exercise for the student.]

8. *P is any point outside a circle of which the centre is O; PM, PN are tangents to the circle, and MN the chord of contact. If OP intersects MN at R, then* OR. OP = OM².



Proof. Since the ∠s OMP, ONP are supplementary, each of them being a rt. ∠,

∴ the quadl. OMPN is cyclic. (*Th. 15.*)

Hence, the $\angle$OMN = the $\angle$OPN (*Th. 11.*)

= the $\angle$OPM. (*Th. 8.*)

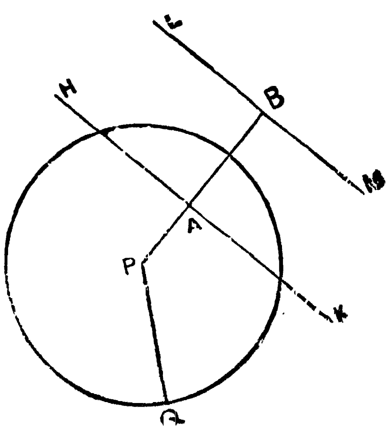
Hence, OM touches at M the $\odot$ circumscribed about the $\triangle$PMR; (*Th. 16, Cor.*)

$\therefore$ OR. OP = OM^2. (*Th. 17, Cor. 1.*)

Note 1. **PM** being = **PN** (Th. 8), the $\perp$ bisector of **MN** passes through **P**; for a similar reason, the $\perp$ bisector of **MN** passes through **O**. Hence **OP** is the $\perp$ bisector of **MN**. Thus, *the chord of contact of tangents drawn to a circle from an external point is bisected perpendicularly by the line joining that point to the centre.*

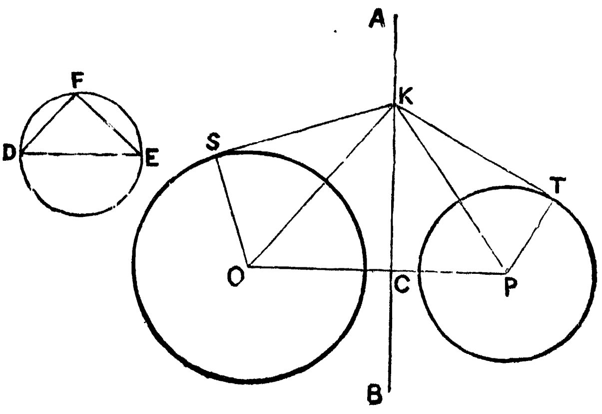
Note 2. If two points are so related to a circle that they lie in a straight line passing through the centre, and the rectangle contained by their distances from the centre is equal to the square on the radius, then each of these points is said to be the **inverse** of the other, with respect to the given circle. Thus, in the preceding diagram, the points **R** and **P** are a pair of *inverse points* with respect to the circle whose centre is **O** and radius = **OM**.

Note 3. If **A** and **B** are a pair of inverse points with respect to a circle whose centre is **P** and radius = **PQ** (that is, if **P**, **A**, **B** are collinear and **PA. PB = PQ^2**), and if through **A** a straight line **HK** be drawn $\perp$ to the line **PAB**, then the point **B** is said to be the **pole** of the line **HK**, and the line **HK**, is said to be the **polar** of the point **B**. Also, if **LM** be drawn through **B** perpendicular to the line **PAB**, then **A** is the *pole* of the line **LM**, and the line **LM** is the *polar* of the point **A**. Thus, in the preceding diagram, the point **P** is the *pole* of the line **MN**, and **MN** is the *polar* of the point **P**.



Note 4. Two circles are said to be **orthogonal**, or to intersect **orthogonally**, when the two tangents at either point of intersection are perpendicular to each other. In the diagram on page 275, **M** is a pt. of intersection of the given ⊙ and the ⊙ about the △**PMR**; also **MP** and **MO**, which are the tangents at **M** to those two ⊙s, respectively, are ⊥ to each other. Hence the given ⊙ and the ⊙ about the △**PMR** intersect *orthogonally*. Similarly, the given ⊙ and the ⊙ about the △**PNR** intersect *othogonally*.

9. O *and* P *are the centres of two given circles, the line* OP *being greater than the sum of their radii. Divide* OP *into two parts such that the difference of the squares on them may be equal to the difference of the squares on the radii of the circles. If* C *be the point of division, and if a straight line be drawn through* C *perpendicular to* OP, *prove that every point on this straight line is such that tangents from it to the two circles are equal.*



(i) Let the ⊙ with centre O have the large radius.

Cons. Take a str. line DE = the radius of the circle (O); on DE as diameter describe a ⊙, and in it place the chord EF = the radius of the circle (P); join DF.

Then the ∠DFE, being in a semi-⊙, is a rt. ∠; and we have $DF^2 = DE^2 - EF^2$

= the difference of the sqs. on the radii.

Now, divide OP at C such that $OC^2 - CP^2$ may be $= DF^2$. (*Bk. II, Sec. IV. Prop. 4.*)

Then C is the reqd. pt.

(ii) Let AB be the str. line drawn through C ⊥ to OP; on AB take *any* pt. K.

Let KS and KT be tangents to the ⊙s (O) and (P) respectively.

To prove that KS = KT.

Proof. Join OS, PT, OK, PK.

Then the ∠s OSK and PTK are right. (*Th. 7.*)

$$\left.\begin{aligned} \text{Now,}\quad OK^2 &= OC^2 + CK^2, \\ \text{and}\quad PK^2 &= PC^2 + CK^2. \end{aligned}\right\}$$

$$\begin{aligned} \therefore\quad OK^2 - PK^2 &= OC^2 - PC^2 \\ &= DF^2 \\ &= OS^2 - PT^2 \end{aligned}$$

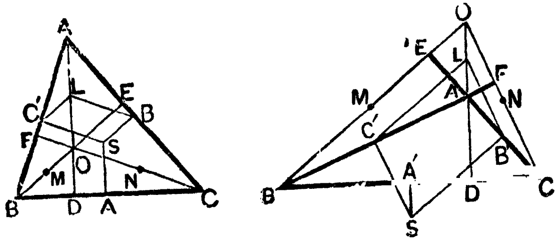
Hence, $OK^2 - OS^2 = PK^2 - PT^2$.

Hence, $KS^2 = KT^2$, and ∴ KS = KT. Q. E. D.

Note 1. The str. line **AB**, which has the property proved above, is callad the **radical axis** of the two given circles.

Note 2. From Note to Cor. 1, Th. 17, it is clear that when two circles intersect, their common secant satisfies the condition of the radical axis, obviously with the exception of that portion which forms the common chord of the two circles.

10 *The distance of each vertex of a triangle from the or. o-centre is double the distance of the circum-centre from the opposite side.*



Let O be the ortho-centre of the $\triangle$ ABC; AD, BE, CF being the $\perp$s from A, B, C on the opposite sides.

Let S be the circum-centre of the $\triangle$ ABC, and A′, B′, C′ the mid-pts. of the sides opposite to A, B, C respectively.

Then SA′, SB′, SC′ are $\perp$s respectively to BC, CA, AB. *(Th. 1.)*

To prove that AO = 2SA′, BO = 2SB′, CO = 2SC′.

Proof. SA′ is ∥ to AD, because both of them are $\perp$ to BC;

Similarly, SB′ is ∥ to BE, and SC′ is ∥ to CF. Let L, M, N be the mid-pts. of AO, BO, CO respectively.

Join LB′, LC′.

Now, B′L passes through the mid-pts. of CA and OA;

$$\left.\begin{array}{l}\therefore \quad B'L \text{ is } \parallel \text{ to } CO \text{ and } \therefore \text{ to } SC'; \\ \text{similarly, } C'L \text{ is } \parallel \text{ to } BO \text{ and } \therefore \text{ to } SB'.\end{array}\right\}$$

Hence the quadl. SL is a par^{m}.;

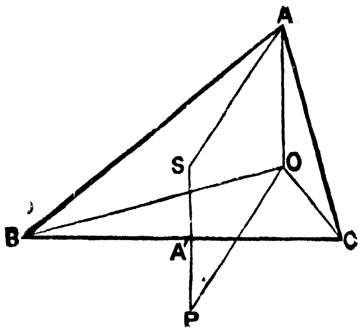
$$\left.\begin{array}{l}\therefore \quad SB' = C'L = \tfrac{1}{2}BO, \\ \text{and } SC' = B'L = \tfrac{1}{2}CO.\end{array}\right\}$$

Similarly, by joining MC′, MA′, (*or* NA′, NB′), it can be shewn that $SA' = \frac{1}{2}AO$. Q. E. D.

Note. If **O** is the ortho-centre of any given △ **ABC**, the following constructions for the circum-centre should be carefully noted. Through **A′**, the mid-pt. of the side opp. to **A**, draw **A′S** ∥ to, and in the same *sense* as, **OA**; make $A'S = \frac{1}{2}$ **OA**. Then **S** is the circum-centre of the given △.

Cor. 1. *If* **O** *is the ortho-centre of any triangle* **ABC**, *then the circum-circles of the four triangles* **ABC, BOC, COA, AOB** *are equal to one another.*

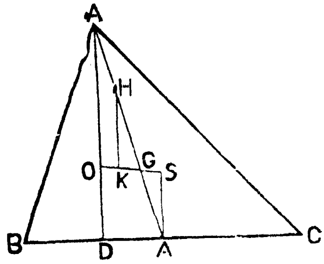
It is easy to see that the pts. A, B, C are the ortho-centres of the △s BOC, COA, AOB respectively. Let S be the circum-centre of the △ ABC, and A′ the mid-pt. of BC. Then SA is ∥ to AO and $= \frac{1}{2}AO$. Produce SA′ to P, making A′P = SA′ (or $\frac{1}{2}$AO);



then P is the circum-centre of the △BOC. Join SA, PO. Now, SP is equal and ∥ to AO; ∴ the quadl. SO is a par^{m}. Hence, SA = PO. Thus, the circum-⊙s of the △s ABC, BOC, having equal radii, are equal to one another.

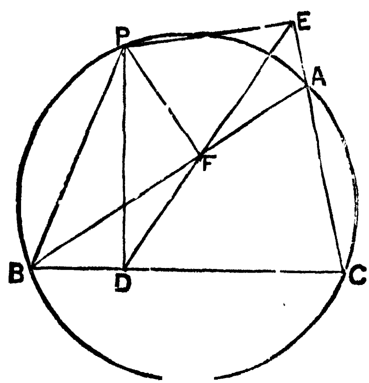
Similarly, the circum-⊙s of the △s COA, AOB are each = the circum-⊙ of the △ABC.

Cor. 2. *The circum-centre, the centroid and the ortho-centre of a triangle are collinear.*



Let S be the circum-centre, and O the ortho-centre, of the △ABC; and let A′ be the mid-pt. of the side BC. Join SA′ and SO; also join AA′ cutting OS in G. Let H, K be the mid-pts. of AG and OG respectively; join HK. Now HK is ∥ to AO and $= \frac{1}{2}$ AO; ∴ HK is equal and ∥ to SA′, and ∴ the quadl. HKA′S is a par^{m}. (*Bk. 1, Th. 22*). But the diagonals of a par^{m}. bisect each other; ∴ A′G = GH $= \frac{1}{2}$ AG. Hence, G is the centroid of the △ ABC, and it lies on the line SO; which proves the corollary.

11. *If perpendiculars be drawn to the sides of a triangle from any point on the circum-circle, then the feet of these perpendiculars lie in one straight line.*



From any pt. P on the circum-⊙ of the △ ABC, let PD, PE, PF be drawn ⊥s to BC, CA, AB respectively.

Join DF, FE.

To prove that DF, FE are in one str. line.

Proof. Join PA, PB.

Since the ∠PDB = the ∠PFB, each being a rt. ∠, the quadl. PFDB is cyclic.

Also, since the ∠s PEA and PFA are supplementary, each being a rt. ∠, the quadl. PFAE is cyclic. (*Th. 15.*)

Now, the quadl. PBCA being cyclic, the ∠ PBC = the ∠PAE ; (*Th.* *14, Cor. 1.*)

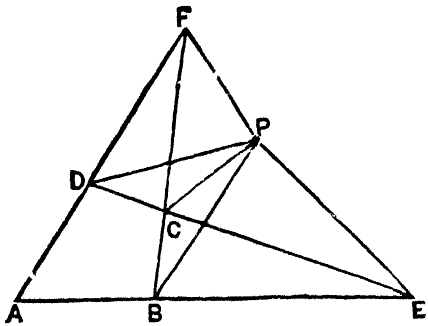
and, the quadl. PFAE being cyclic, the ∠PAE = the ∠PFE. (*Th. 11.*)

Hence, the ∠FBD = the ∠PFE.

But, the quadl. PFDB being cyclic, the ∠s PBD and PFD are supplementary ; ∴ the ∠s PFE and PFD are supplementary, and ∴ DF, FE are in one str. line. Q. E. D.

Note. The line DFE is called the **pedal line** of the point P with respect to the triangle ABC.

12. *The circum-circles of the four triangles formed by four given straight lines, of which no two are parallel, have one point common to them all.*



Let the four str. lines AB, DC, AD, BC form the four △s ABF, DCF, ADE, BCE. C is one pt. of intersection of the ⊙s DCF, BCF ; let their second pt. of intersection be P.

To prove that P is a pt. on the ⊙ABF and also on the ⊙ ADE.

Proof. Join CP, BP, DP.

The quadl. BCPE being cyclic, the △ PBE = the ∠PCE ; (*Th. 11.*)

and the quadl. DCPF being cyclic. the ∠ PCE = the ∠ PFD. (*Th. 14, Cor. 1.*)

Hence, the ∠PBE = the ∠PFA,

and ∴ the quadl. ABPF is cyclic. (*Th. 15, Cor.*)

Thus, P is a pt. on the ⊙ ABF.

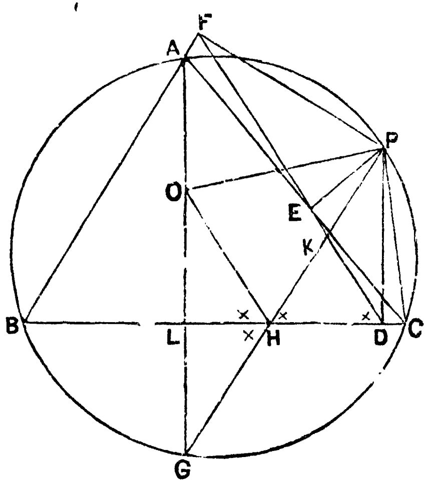
In the same way, it can be shewn that P is a pt. on the ⊙ ADE.

Note 1. *If the four given straight lines are such that the quadl.* ABCD *is cyclic, then the three pts.* F, P, E *are collinear.*

For, in that case, the ∠FPC = ∠CDA = the ∠CBE ; ∴ the ∠s FPC and EPC are together = the ∠s CBE and EPC = two rt. ∠s. Hence, FP and PE are in the same str. line.

Note 2. *If from* P *the* ⊥s PH, PK, PL, PM *be drawn to the lines* ABE, DCE, BCF, ADF *respectively, then the four pts.* **H, K, L, M** *are collinear.* Since **P** is a pt. on the circum-⊙ of the △**BCE**, and **PH, PK, PL** are the ⊥s from **P** on the sides of a triangle, ∴ the three pts. **H, K, L** lie in one str. line, (Prop. 12). Similarly, the three pts. **K, L, M** are in one str. line. Thus, the str. line that passes through **K** and **L** also passes through **H** and **M** ; which shews that all the four pts. lie in one str. line.

13. *The line joining the ortho-centre O of a triangle ABC to any point P on the circum-cirble is bisected by the pedal line of P with respect to the triangle.*



In the above diagram, let DEF be the pedal line of that pt. P.

To prove that OP is bisected by the line DEF.

Proof. Produce AO to meet BC in L and the ⊙ in G.

Join PG, cutting BC at H, and the pedal line at K.

Join OH, PC. The line AOL is ⊥ to BC ;

$\therefore$ OL = LG. (*Prop.* 2, *Cor.* 2.)

Hence the $\triangle$s OLH, GLH are congruent ;
and $\therefore$ the $\angle$ LHG = the $\angle$ LHO.

Now, the quadl. PEDC being cyclic,

the $\angle$ EDH = the $\angle$EPC

= the complement of the $\angle$ PCA

= the complement of the $\angle$ PGA

= the $\angle$LHG

= the $\angle$LHO ;

$\therefore$ DF is $\parallel$ to HO.

Again, the $\angle$KDH = the $\angle$LGH (*Proved above*)

= the $\angle$KHD ;

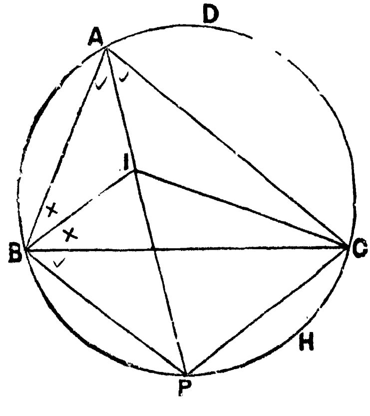
$\therefore$ the $\angle$ KDP = the $\angle$KPD.

Hence, KH = KD = KP.

Thus, K is the mid-pt. of the side PH of the $\triangle$ OPH, and the line DKF is $\parallel$ to the side HO ;

$\therefore$ the side OP is bisected by the line DEF. Q.E.D.

14. *If triangles standing on the same base, on the same side of it, have equal vertical angles, then their in-centres lie on the circumference of a circle whose centre is a known point on the perpendicular bisector of the given base.*



(i) Let BAC be one of the $\triangle$s standing on the given base BC and having a given vertical $\angle$.

Let P be the mid-pt. of the arc BHC of the circum-$\odot$ BAC; join PA.

Then the $\angle$ FAB = the $\angle$ PAC; (*Th. 10, Cor. 2.*)

$\therefore$ the in-centre of the $\triangle$BAC lies on PA. (*Prob. 5.*)

Let I be the in-centre; join BI, CI.

AB, BC, CA being tangents to the in-$\odot$, BI, CI are the bisectors of the $\angle$s B, C respectively. (*Th. 8.*)

Join PB, PC.

Now, the $\angle$ PBC = the $\angle$ PAC = the $\angle$ IAB;

$\therefore$ the $\angle$ PBI = the $\angle$ IAB + the $\angle$ IBA
= the $\angle$ BIP.

Hence, PI = PB; and similarly,
PI = PC.

Thus PB = PI = PC. ... ... (α)

(ii) Let D be the vertex of *any other* $\triangle$ standing on the base BC, on the same side of it as A, and having the vartical $\angle$ BDC = the $\angle$ BAC.

Then the pt. D must be on the arc BAC of the circum-$\odot$ of the $\triangle$BAC. (*Th. 12.*)

Hence, if I$'$ be the in-centre of the $\triangle$BDC, it can be shewn, as before, that PI$'$ = PB. ... ... (β)

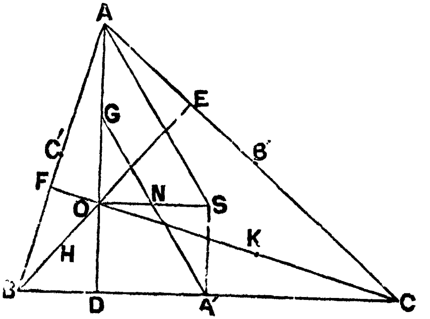
Hence, from (α) and (β), if a $\odot$ be described with P as centre and PB or PC as radius, it will pass through the in-centres of *all* the $\triangle$s of which the $\triangle$BAC is one.

Now, since PB = PC, the pt. P is evidently on the $\perp$ bisector of BC;

Thus, P is the pt. where the ⊥ bisector of BC meets the arc BHC of the common circum-⊙ of the △s considered, and is ∴ a *known* pt. Q. E. D.

Note. It should be carefully observed that if the base and the vertical angle of a triangle be given, the triangle admits of various positions. Considering the positions on one side of the base only, it is clear that whatever may be the position, the circum-circle is the same; for, the circle that passes through the extremities of the base and one position of the vertex, also passes through *every other* position of the vertex. Hence, when the base and the vertical angle of a triangle are given, we may say that the circum-circle is *fixed* (*i.e. not different* for different positions of the triangle); and ∴ the circum-centre is *fixed* and the circum-radius is of *constant* magnitude.

15. *In any triangle, the middle points of the sides, the feet of the perpendiculars from the vertices on the opposite sides, and the middle points of the lines joining the ortho-centre to the vertices are concyclic.*



In the above diagram, O is the ortho-centre and S the circum-centre of the △ABC; A′, B′, C′ are the mid-pts. of

the sides opp. to A, B, C; AD, BE, CF are the $\perp$s from A, B, C on the opp. sides; and G, H, K are the mid-pts. of the lines OA, OB, OC.

To prove that the nine points A′, B′, C′, D, E, F, G, H, K are concyclic.

Proof. Join SA, SA′, SO and GA′ cutting SO at N.

Now, SA′ is ∥ to AO and $= \frac{1}{2}$ AO;

∴ SA′ is equal and ∥ to GO.

Hence G, O, A′, S are the consecutive angular pts. of a parallelogram. (*Bk. I, Th. 22.*)

Hence GA′ and OS bisect each other at N.

Thus GA′ is bisected at the mid-pt. of OS; and similarly, HB′ and KC′ are each bisected at the mid-pt. of OS.

Now, GDA′ being a rt. ∠, the ⊙ described on GA′ as diameter passes through D. (*Th. 13, Cor. 2.*)

∴ ND = NG = NA′, each of them being a radius of this ⊙.

Again, since SA′ is equal and ∥ to AG,

∴ SA = A′G. (*Bk. I, Th. 22.*)

Hence, if R stands for "the circum-radius of the △ ABC,"

we have ND = NG = NA′ = $\frac{1}{2}$R. (SA being R.)

$$\left.\begin{array}{ll}\text{Similarly, } NE = NH = NB' = \tfrac{1}{2}R, \\ \text{and } \quad NF = NK = NC' = \tfrac{1}{2}R.\end{array}\right\}$$

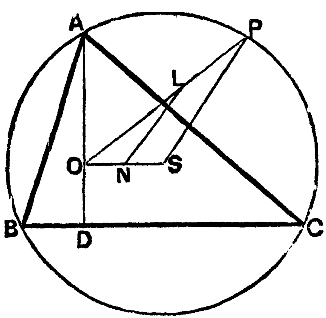
Thus, the nine pts. D, G, A′, E, H, B′, F, K, C′ are each at a distance = $\frac{1}{2}$ R from N, the mid-pt. of OS;

and ∴ they all lie on the O^{ce} of the ⊙ whose centre is the mid-pt. of O6 and radius = $\frac{1}{2}$ R.

Note. The circle passing through the mid-points of the sides of a triangle is called the **Nine-point circle** of the triangle, from its property of passing through *nine* particular points connected with the triangle. The centre of the nine-point circle may also be conveniently called the "nine-point centre."

Cor. 1. *The nine-point circle of each of the triangles* BOC, COA, AOB *is the same as that of the triangle* ABC.

Cor 2. *If* P *is any point on the circum-circle of the triangle* ABC, *the mid-point of* OP *is on the nine point circle.*

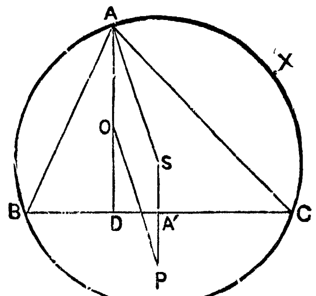


If L is the mid-pt. of OP, join NL. Then NL is ∥ to SP, and = $\frac{1}{2}$ SP = $\frac{1}{2}$ R. Thus NL is = a radius of the nine-pt. ⊙, and ∴ the pt. L lies on that ⊙.

Cor. 3. *Given the base and the vertical angle of a triangle, as the triangle changes its position the centre of the nine-point circle moves along the circumference of a circle whose centre is the mid-point of the given base.*

In the figure of the proposition, suppose XBC to be *any other* position of the triangle, the base BC remaining fixed and the vertical ∠ BXC being = the ∠ BAC. Then X must be on the arc BAC of the circum-⊙ of the △ ABC. Now, if N′ be the nine pt. centre of the △XBC, we must have N′A′ = $\frac{1}{2}$SX = $\frac{1}{2}$SA. Thus, N and N′ are both at the same distance from A′; which proves the corollary.

16. *Given the base and the vertical angle of a triangle, prove that, as the triangle changes its position, (i) the ortho-centre moves along the circumference of a circle of which the centre is a known point on the perpendicular bisector of the given base, and (ii) the centroid also moves along the circumference of another circle of which the centre is a known point on the same bisector.*



(i) Let ABC be a △ standing on the given base BC and having its vertical ∠ of the given magnitude.

Let AD be the ⊥ on BC, O the ortho-centre, S the circum-centre, and A′ the mid-pt. of BC.

Join SA′ and produce it to P making A′P = SA′;

join SA, PO.

Now, SA′ is ∥ to AO and = ½ AO ;

∴ SP is equal and ∥ to AO.

Hence, PO = SA. (*Bk. I. Th. 22.*)

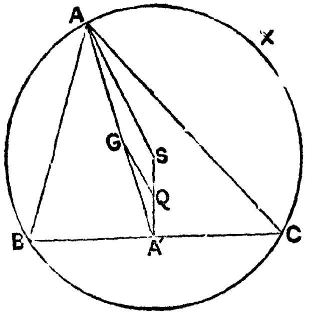
Let XBC be *any other* position of the $\triangle$, the base BC remaining fixed and the vertical $\angle$BXC being = the $\angle$BAC.

Then X must be on the arc BAC of the cirnum·$\odot$ of the $\triangle$ ABC.

Now, if O′ be the ortho-centre of the $\triangle$ XBC, it can be shewn, as before, that PO′ = SX.

Thus, PO = PO′ (each of them being = SA), which shews that O and O′ are on the $\bigcirc^{ce}$ of a $\odot$ of which the centre is the pt. P and the radius = SA.

Also, since A′P = SA′, P is obviously a known pt. on the $\perp$ bisector of BC.



(ii) Let ABC be a $\triangle$ standing on the given base BC and having the vertical $\angle$BAC of the given magnitude.

Let A′ be the mid-pt. of BC, and G the centroid ; then G is a pt. on AA′ such that AA′ = 3GA′.

Let S be the circum-centre ; join SA′, SA.

Draw GQ $\parallel$ to AS, meeting SA′ in Q.

Now, since AS is   to GQ and $AA' = 3A'G$,

$\therefore \quad AS = 3GQ.$ *(Bk. I. Sec. VIII, Prop. 5.)*

$\therefore \quad GQ = \frac{1}{3}AS.$

Let XBC be *any other* position of the $\triangle$, the base BC remaining fixed and the vertical $\angle BXC$ being = the $\angle BAC$.

The X must be on the arc ABC of the circum-$\odot$ of the $\triangle$ ABC.

Now, if G' be the centroid of the $\triangle$ XBC, it can be shewn, as before, that $G'Q = \frac{1}{3}XS$.

Thus, $GQ = G'Q$ (each of them being $= \frac{1}{3}SA$), which shews that G and G' are on the $\bigcirc^{ce}$ of a $\odot$ of which the centre is the pt. Q, and the radius $= \frac{1}{3}SA$.

Also, since $QA' = \frac{1}{3}SA'$, Q is obviously a known pt. on the $\perp$ bisector of BC.

EXERCISE (27).

1. Shew that the middle points of the chords of a circle which pass through a given point all lie on the circumference of another circle.

2. If a quadrilateral be circumscribed about a given circle, prove that the sum of any pair of opposite sides is equal to the sum of the other pair. Hence prove that any parallelogram circumscribed about a circle is a rhombus.

3. Prove that the triangle formed by joining the three points in which the in-circle of a triangle meets the sides, is acute-angled.

4. The diagonals AC, BD of a parallelogram ABCD intersect at O. Shew that the circum-circles of the triangles AOB, COD touch each other at O.

5. Shew that the greatest quadrilateral that can be inscribed in a circle is a square,

6. On the sides BC, CA, AB of a given triangle ABC, any three points D, E, F respectively are taken. Prove that the circum-circles of the triangles EAF, FBD, DCE meet in a point.

7. If the in-circle of a triangle ABC touches the sides BC, CA, AB at D, E, F respectively, prove that BC + AF = CA + BD = AB + CE = the semi-primeter of the triangle ABC.

8. With the vertices A, B, C of a triangle as centres construct three circles each of which shall touch the other two.

9. Prove that the sum of any three alternate angles of a hexagon inscribed in a circle is equal to four right angles.

10. AB is the common chord of two circles, and P is any point on the circumference of one of them so that the straight lines PAQ, PBR meet the circumference of the other in Q and R. Prove that the arc QR is of the same magnitude for different positions of P.

11. The opposite sides of a cyclic quadrilateral are produced to meet in P and Q, and about the triangles so formed without the quadrilateral circles are described, which meet in R. Prove that the points P, R, Q are collinear.

12. ABC is a triangle; BCA′, CAB′, ABC′ are equilateral triangles so that the points (A, A′), (B, B′), (C, C′) are on opposite sides of BC, CA, AB respectively. Prove that the circum-circles of the three equilateral triangles meet at a point which is also the point of concurrence of the lines AA′, BB′, CC′.

13. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected, prove that the line joining the points of bisection is trisected by the sides,

14. P is any point on the circum-circle of a given triangle ABC; PA′, PB′, PC′ are chords of the circle perpendicular respectively to BC, CA, AB. Shew that the triangles ABC, A′ B′ C′ are congruent.

15. If S be the circum-centre of a triangle ABC, and if the perpendicular from S on BC meets the circum-circle in K and L, L being on the same side of BC as A, prove that AK and AL bisect the interior and exterior angles at A. Also, if I be the in-centre, prove that K is the circum-centre of the triangle BIC.

16. ABC is a triangle; O is the point of intersection of the perpendiculars AD, BE, CF upon the sides of the triangle; G, H, K are the middle points of the sides, and L, M, N the middle points of the lines OA, OB, OC. Prove that each of the angles LHG, LKG is a right angle; and hence prove that the circle passing through the points G, H, K also passes through the six points (L, D), (M, E), (N, F).

17. Given one angle, of a triangle, the side opposite to it, and the sum of the other two sides, construct the triangle.

18. ABC is any triangle, inscribed in a circle, and AP, BQ are chords of the circle parallel to BC, CA respectively. Prove that PQ is parallel to the tangent at C.

19. AB is the diameter of a semi-circle, D and E are any two points on its circumference. AE, BD intersect at M; and AD, BE produced intersect at L. Prove that LM produced cuts AB at right angles.

20. BC is a given arc of a circle whose centre is O; A is any point in BC. AD, AE are drawn perpendiculars to OB, OC. Prove that the line DE is of constant length.

21. Prove that an equiangular polygon inscribed in a circle has its alternate sides equal. Hence, shew that an equiangular pentagon inscribed in a circle is also equilateral.

22. If any number of triangles be on the same base, on the same side of it, and have equal vertical angles, prove that the bisectors of the vertical angles are concurrent.

23. If two circles intersect and if one of them passes through the centre of other, prove that the tangents to the latter at the points of intersection will meet on the former.

24. If P be any point on the circle circumscribed about a given equilateral triangle ABC, prove that one of the lines PA, PB, PC is equal to the sum of the other two.

25. O is the circum-centre of a triangle ABC, and D, E, F are the feet of the perpendiculars from A, B, C on the opposite sides. Shew that OA, OB, OC are respectively perpendiculars to EF, FD, DE.

26. Right-angled triangles are described on the same hypotenuse. Shew that the locus of the centres of the in-circles is a quarter of the circumference of a circle of which the common hypotenuse is a chord.

27. Construct a triangle, having given one side and the radii of the in-circle and circum-circle.

28. ABCDE is a regular pentagon inscribed in a circle, and AC, BD intersect at O. Prove that AO = DO, and that $BC^2 = AC. CO$.

29. ACDB is a semi-circle, AB being the diameter, and the two chords AD, BC intersect at E. If a circle be circumscribed about the triangle CDE, prove that it will cut the former at right angles.

30. If I be the in-centre, and I_1, I_2, I_3, the ex-centres of a given triangle, prove that I is the ortho-centre of the triangle $I_1I_2I_3$. Hence shew that the circum-circles of the four triangles II_2I_3, II_3I_1, II_1I_2 and $I_1I_2I_3$ are equal to one another.

31. The diagonals of a given quadrilateral ABCD intersect at O. Shew that the centres of the circles circumscribed about the triangles OAB, OBC, OCD, ODA are at the angular points of a parallelogram.

32. If a line BC of constant length have its extremities on two fixed str. lines AX, AY, prove that the circum-radius of the $\triangle ABC$ is of the same length for all positions of BC. Hence prove that, as BC changes its position, the circum-centre of the $\triangle ABC$ moves on the $\bigcirc^{ce}$ of a $\odot$.

33. In the preceding example, shew that the distance of the circum-centre from the line BC is the same for all positions of BC. Hence prove that, as BC changes its position, the ortho-centre of the triangle ABC moves on the circumference of a circle.

34. ABC is an equilateral triangle inscribed in a circle whose centre is O, and BO is produced to meet the circumference in D. Prove that the arc AD is one-sixth of the whole circumference. Hence prove that $AD = AO$, and that $AB^2 = 3OA^2$.

35. PM, PN are tangents to a circle, of which the centre is O, and MN the chord of contact. PEF is any straight line cutting the circle at E, F. OS is drawn perpendicular to EF and produced to meet MN produced in Q. Prove that $OS.OQ = (\text{radius})^2$, and hence prove that QE, QF are tangents to the circle.

36. Through a given point without a given circle draw a straight line so that the part intercepted by the circumference may be equal to a given straight line not greater than the diameter.

37. The internal and external bisectors of the angle A of a triangle meet the base BC in E, E′ and the circum-circle in D and D′. Prove that D is the ortho-centre of the triangle EE′D′.

38. From S, the circum-centre of the triangle ABC, perpendiculars SA′, SB′, SC′ are drawn to the sides, and these perpendiculars are produced to P, Q, R respectively so that SA′ = A′P, SB′ = B′Q and SC′ = C′R. Prove that S is the ortho-centre of the triangle PQR, and hence shew that the △s ABC, PQR have the same nine-point circle.

39. Construct a triangle, having given the ortho-centre, the circum-centre and one angular point.

40. AB and AC are two given str. lines in which B and C are two given pts. BD is drawn ⊥ to AC, and DE ⊥ to AB; in like manner CF is drawn ⊥ to AB, and FG to AC. Prove that EG is ∥ to BC.

41. If the centres of two circles which touch each other externally be fixed, prove that the external common tangents of the two circles will also touch the circle of which the straight line joining the fixed centres is the diameter.

42. AB, AC are the equal sides of an isosceles triangle ABC. P is a point on AB, and Q a point on AC produced such that BP = CQ. Prove that, for all positions of PQ, the circum-circle of the triangle APQ passes through a fixed point on the bisector of the angle BAC.

43. ABC is a triangle, and BE, CF are the perpendiculars from B and C upon the opposite sides. If K be the mid-pt. of BC, prove that KF = KE, and that each of the ∠s KFE, KEF is equal to the ∠A.

44. If two circles touch each other internally, prove that any chord of the greater circle which touches the smaller is divided at the point of contact into segments which subtend equal angles at the point of contact of the two circles.

45. PA, PB, PC are any three chords of a circle. Prove that the circles described on these chords as diameters will intersect again in three points which are collinear.

46. A series of circles touch a fixed straight line at a fixed point. Shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

47. AD, BE, CF are the perpendiculars of the triangle ABC. Prove that the feet of the $\perp$s from D on AB, AC, BE, CF are collinear.

48. O is the ortho-centre of a $\triangle$ ABC, and P, Q, R the circum-centres of the $\triangle$s BOC, COA, AOB respecttively. Prove that O is the circum-centre of the $\triangle$ PQR. Prove also that each of the quadls. OPCQ, OQAR and ORBP is a rhombus, and that the $\triangle$s ABC, PQR are congruent.

49. Having given the base and the vertical angle of a triangle, prove that the nine-point circle touches a fixed circle whose radius is equal to that of the circum-circle.

50. A man standing on the deck of a ship in the middle of a calm ocean looks around with a telescope. Shew that the distance from him of the farthest object he can see on the surface of the ocean is approximately $\sqrt{\frac{1}{8}}$ *mile*, where 1 *inch* is the height of the observer's eye above the water, assuming the radius of the earth to be 4000 miles.

BOOK IV.

Proportion : Similar Triangles.

SECTION I.

FUNDAMENTAL IDEAS AND DEFINITIONS.

1. A straight line, of an arbitrarily chosen length, which is used for the puppose of ascertaining the lengths of other straight lines, is called the **unit of length.** We are said **to measure** a straight line when we ascertain the number of times that it contains the unit of length.

2. The area of a square of which a side is equal to the unit of length is called the **unit of area.** We are said **to measure** the area of a plane figure when we ascertain the number of times that this area contains the unit of area.

3. An angle, of an arbitrarily chosen size, which is used for the purpose of ascertaining the sizes of other angles, is called the **unit of angle.** We are said **to measure** an angle when we ascertain the number of times that it contains the unit of angle.

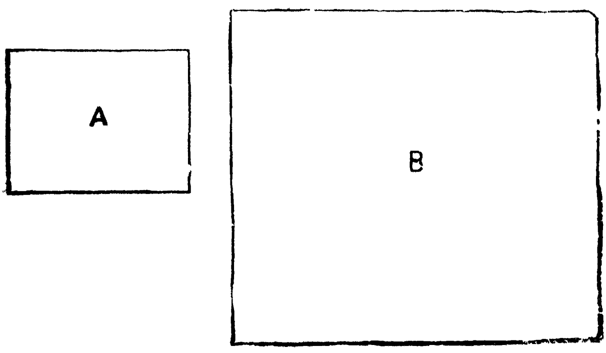
4. Any thing that can be measured is called a **magnitude.** Clearly therefore, a straight line, an angle and the area of a plane figure are all *magnitudes.*

Note. This is the more extended sense in which the word "magnitude" is used. For other meanings of the term see Art. 5, Sec. 1, Book I.

5. The **numerical measure** of any given magnitude is the number of times that it contains the unit of its kind. Thus, if one inch be taken as the unit of length, the *numerical measure* of a straight line that contains six complete inches with a remainder which is two-thirds of an inch long, is $6\frac{2}{3}$.

Note. The number of times that any given magnitude contains the unit of its kind, is said to **represent** that magnitude. Thus, if one foot be taken as the unit of length, a straight line which is 9 inches long will be *represented* by the number $\frac{3}{4}$.

6. If two magnitudes A and B be such that B can be divided into parts each of which is equal to A, then A is said to be a **measure** of B, and B is said to be a **multiple** of A. Thus, in the following diagram, if the areas of the rectangles A and B be $\frac{3}{4}$ of a square inch and $3\frac{3}{4}$

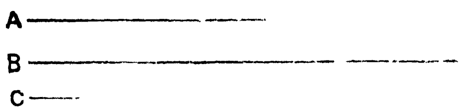


square inches respectively, then it is easy to see that B can be divided into five parts each of which is equal in area to A; hence, A is a *measure* of B, and B is a *multiple* of A.

Note 1. In the above illustration, the expression "**A** is a measure of **B**" means *the area of* **A** *is a measure of the area of* **B.** Similar to the expression "**B** is a multiple of **A**" means *the area of* **B** *is a multiple of the area of* **A.**

Note 2. When **B** is divisible into parts each of which is equal to **A**, it may be said that **B** *contains* **A** *an integral number of times.*

Note 3. *If two magnitudes have a common measure, they have also a common multiple. Let* A, B *and* C *be three straight lines such that* C *is a common measure of* A *and* B.



Then, we must have

$$\left.\begin{array}{ll} & \text{A} = m \text{ times C} \\ \text{and} & \text{B} = n \text{ times C} \end{array}\right\}, \text{ where } m \text{ and } n \text{ are two integers.}$$

$$\left.\begin{array}{ll} \text{Hence,} & n\,\text{A} = mn.\text{C} \\ \text{and} & m\text{B} = mn\,\text{C} \end{array}\right\};$$

$$\text{and } \therefore\ n\text{A} = m\text{B}.$$

Thus, the straight line which is n times A is also that which is m times B, and is therefore a common multiple of A and B.

7. If two magnitudes of the same kind have a common measure they are said to be **commensurable** ; if they have no common measure they are said to be **incommensurable.**

Note. The side and the diagonal of a square are a pair of ***incommensurable*** magnitudes. For, if the length of the side be a inches, the length of the diagonal must be $a\sqrt{2}$ inches *i.e.*, ($a \times 1{\cdot}4142135......$) inches.

Hence, the diagonal $= a$ inches $\times 1{\cdot}4142135\ldots\ldots$

$$= \left(\frac{a \text{ inches}}{100}\right) \times 141{\cdot}42135\ldots\ldots\ldots ;$$

which shews that one-hundredth part of the side, which of course is a measure of the side, is not contained an integral number of times in the diagonal, and is therefore *not* a measure of the diagonal.

Again, the diagonal $= \left(\frac{a \text{ inches}}{215}\right) \times (215 \times 1{\cdot}414\ldots\ldots)$

$$= \left(\frac{a \text{ inches}}{215}\right) \times 304{\cdot}011\ldots\ldots ;$$

which shews that one 215th part of the side, which ofcourse is a measure of the side, is *not* a measure of the diagonal.

And so on. Clearly therefore *no measure* of the side is a measure of the diagonal, and consequently the side and the diagonal *have no common measure.*

8. If two magnitudes are incommensurable they may still be regarded as commensurable for all *practial purposes.* For instance, take a square of which the side is ten inches in length ; then

(i) The diagonal $=$ (10 inches) $\times$ $1{\cdot}414213562\ldots\ldots$

$$= \left[\frac{10 \text{ inches}}{1{,}00{,}000}\right] \times 141421{\cdot}3562\ldots\ldots$$

$=$ (one ten-thousandth of an inch) $\times$ $141421{\cdot}3562\ldots\ldots$

$=$ 141421 times (one ten-thousandth of an inch) $+$ $\cdot3562\ldots\ldots\ldots$ of (one ten-thousandth of an inch) ;

which shews that if the portion of the diagonal which is 141421 times (one ten-thousandth of an inch) be taken as the diagonal, leaving out the remainder which is *less than one ten-thousandth of an inch*, then the side and the diagonal may be regarded as *commensurable*, the common measure being one ten-thousandth of an inch.

(ii) The diagonal $= \left(\dfrac{10 \text{ inches}}{10{,}00{,}000}\right) \times 1414213{\cdot}562\ldots\ldots$

$=$ (one hundred-thousandth of an inch) $\times\ 1414213{\cdot}562\ldots\ldots$

$=$ 1414213 times (one hundred-thousandth of an inch) $+\ {\cdot}562\ldots\ldots$ of (one hundred-thousandth of an inch) ;

which shews that if the portion of the diagonal which is 1414213 times (one hundred-thousandth of an inch) be taken as the diagonal, leaving out the remainder which is *less than one hundred-thousandth of an inch*, then the side and the diagonal may be regarded as *commensurable*, the common measure being one hundred-thousandth of an inch.

(iii) The diagonal $= \left[\dfrac{10 \text{ inches}}{10{,}000{,}000}\right] \times 14142135{\cdot}62\ldots\ldots\ldots\ldots$

$=$ (one millionth of an inch) $\times\ 14142135{\cdot}62\ldots\ldots\ldots\ldots$

$=$ 14142135 times one millionth of an inch $+\ {\cdot}62\ldots\ldots$ of one millionth of an inch ;

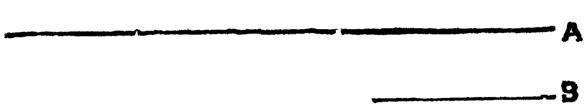
which shews that if the portion of the diagonal which is 14142135 times one-millionth of an inch be taken as the diagonal, leaving out the remainder which is *less than one-millionth of an inch*, then the side and the diagonal may be regarded as *commensurable*, the common measure being one-millionth of an inch.

And so on

From the above it is clear that if two magnitudes are incommensurable they may still be regarded as commensurable, the common measure being a *very small* measure of one of them ; and the smaller this measure is, the smaller is the consequent error. In other words, if two magnitudes are incommensurable, one of them may be regarded as a *very large* multiple of a *very small* measure of the other, the consequent error being too small for *practical* purposes.

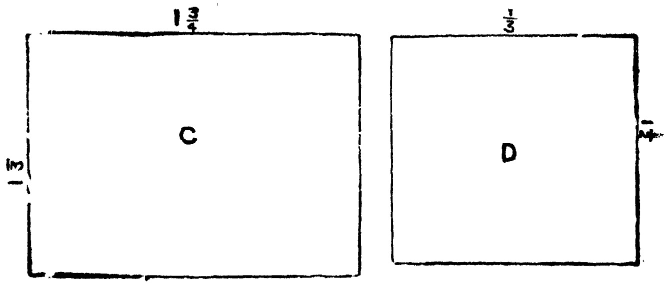
9. The **ratio** of one magnitude to another of the same kind is the number which expresses what multiple or what fraction the former is of the latter.

(i) If A and B are two straight lines whose lengths are 3 inches and 1 inch respectively.



then the ratio of A to B is 3 ; and the ratio of B to A is $\frac{1}{3}$.

(ii) If C and D are two rectangles whose areas are $2\frac{1}{3}$ and $1\frac{2}{3}$ square inches respectively,



then the ratio of C to D is $\frac{2\frac{1}{3}}{1\frac{2}{3}}$, which is $=\frac{7}{5}$; and the ratio of D to C is $\frac{1\frac{2}{3}}{2\frac{1}{3}}$, which is $=\frac{5}{7}$.

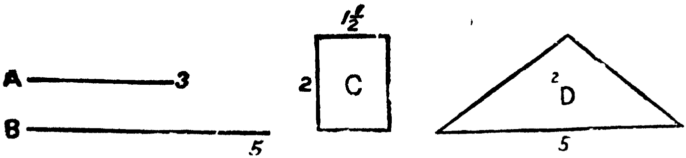
(iii) Generally speaking, if A and B are two magnitudes of the same kind, whose *numerical measures* referred to one and the same unit are a and b respectively, then the ratio of A to B is $\frac{a}{b}$ and the ratio of B to A is $\frac{b}{a}$.

Note 1. The symbol $\frac{A}{B}$ stands for "the ratio of A to B." Hence, "the ratio of A to B is $\frac{a}{b}$" is briefly written as $\frac{A}{B}=\frac{a}{b}$, "The ratio of A to B" is also written as A : B.

Note 2. "The ratio of A to B is $\frac{a}{b}$" and "A and B are in the ratio of a to b" are equivalent expressions.

Note 3. When we consider the ratio of A to B, A is called the **antecedent** (or the first term) and B the **consequent** (or the second term) of the ratio. When the ratio of B to A is considered, B is the ***antecedent*** and A the *consequent*.

Note 4. If **A** and **B** are two straight lines whose numerical measures referred to one and the same unit of length are **3** and **5** respectively; and if **C** and **D** are two areas of which the numerical measures referred to one and the same unit of area are respectively



3 and 5; then the ratio of **A** to **B** is $\frac{3}{5}$, and also the ratio of **C** to **D** is $\frac{3}{5}$. Hence, *the ratio of* **A** *to* **B** *is equal to that of* **C** *to* **D**. This is briefly expressed by the notation $\frac{A}{B} = \frac{C}{D}$.

10. Four magnitudes A, B, C, D are said to be **proportional (or in proportion)** when $\frac{A}{B} = \frac{C}{D}$, *i.e.*, when the ratio of A to B is equal to that of C to D.

Note 1. (i) $\frac{A}{B} = \frac{C}{D}$, (ii) $A : B = C : D$, and (iii) $A : B :: C : D$ are equivalent expressions and read as "**A** is to **B** as **C** is to **D**."

Note 2. Each of the above three expressions is called a **proportion.**

Note 3. In a proportion such as $A : B = C : D$, **A** and **D** are called the **extremes**, and **B** and **C** the **means.**

Note 4. The word "proportional" is also used as a noun; thus, in the proportion $A : B :: C : D$, **D** is said to be the **fourth proportional** to **A**, **B** and **C**.

Note 5. In a proportion, terms occupying similar positions, *i.e.*, terms which are both antecedents or both consequents of the ratios, are said to be **corresponding** or **homologous terms.**

11. Magnitudes A, B, C, D of the same kind are said to be in **continued proportion** when $\frac{A}{B} = \frac{B}{C} = \frac{C}{D}$, *i.e.*, when the ratio of A to B, the ratio of B to C, the ratio of C to D, are all equal

12. When three magnitudes A, B, C of the same kind are in continued proportion, *i.e.*, when

$$\frac{A}{B} = \frac{B}{C},$$

B is said to be the **mean proportional** to A and C: and C is said to be the **third proportional** to A and B.

13. If three magnitudes A, B, C of the same kind be such that the ratio of B to A is equal to that of C to A, then B = C. For, let a, b, c be the numerical measures of A, B, C referred to one and the same unit; then, by hypothesis,

$$\frac{b}{a} = \frac{c}{a},$$

and $\therefore b = c$. Hence B = C.

Thus, **magnitudes which have the same ratio to the same magnitude are equal to one another.**

Note. Conversely, equal magnitudes have the same ratio to the same magnitude.

14. If four magnitudes A, B, C, D be such that the ratio of A to B is equal to that of C to D, then the ratio of B to A is also equal to that of D to C. For, if a, b be the numerical measures of A and B referred to a unit of the class to which A and B belong, and if c, d be the numerical

measures of C and D referred to a unit of the class to which C and D may belong; then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d}$$

$$\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d};$$

$$\text{or,} \qquad \frac{b}{a} = \frac{d}{c}; \text{ whence } \frac{B}{A} = \frac{D}{C}.$$

Thus, **if four magnitudes be proportional, they are also proportional when taken inversely.**

15. If four magnitudes A, B, C, D *of the same kind* be such that the ratio of A to B is equal to that of C to D, then also is the ratio of A to C equal to that of B to D. For, if a, b, c, d be the numerical measures of A, B, C, D respectively, referred to one and the same unit, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c};$$

$$\text{or,} \qquad \frac{a}{c} = \frac{b}{d}; \text{ whence } \frac{A}{C} = \frac{B}{D}.$$

Thus, **if four magnitudes of the same kind be proportional, they are also proportional when taken alternately.**

16. If four magnitudes A, B, C, D be such that the ratio of A to B is equal to that of C to D, then also is the ratio of A + B to B equal to that of C + D to D. For, if

a, b be the numerical measures of A and B referred to a unit of the class to which A and B belong, and if c, d be the numerical measures of C and D referred to a unit of the class to which C and D may belong, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \quad \frac{a}{b} + 1 = \frac{c}{d} + 1;$$

or, $$\frac{a+b}{b} = \frac{c+d}{d}; \text{ whence } \frac{A+B}{B} = \frac{C+D}{D}.$$

Thus, *if four magnitudes be proportional, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.* This inference is spoken of as **componendo**.

Note 1. If A be greater than B, and C greater than D, it is easy to prove that $\frac{A-B}{B} = \frac{C-D}{D}$.

This inference is spoken of as **Dividendo**.

Note 2. Since $\left.\begin{array}{l} \frac{a+b}{b} = \frac{c+d}{d} \\ \text{and } \frac{a-b}{b} = \frac{c-d}{d} \end{array}\right\}$ $\therefore \quad \frac{a+b}{a-b} = \frac{c+d}{c-d}$, whence $\frac{A+B}{A-B} = \frac{C+D}{C-D}$.

This inference is spoken of as **Componendo and Dividendo.**

17. If the ratios of A to B, C to D, E to F be equal to one another, where A, B, C, D, E, F are all magnitudes *of the same kind*, then each of these ratios is equal to the ratio of A + C + E to B + D + F. For, if a, b, c, d, e, f be

the numerical measures of A, B, C, D, E, F respectively, referred to one and the same unit, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

Taking each of these fractions $= k$, we have

$$a = bk,\ c = dk,\ e = fk;$$

$$\therefore \quad a+c+e = k(b+d+f);$$

$$\therefore \quad k = \frac{a+c+e}{b+d+f},$$

$$i.e., \quad \frac{a}{b} \text{ or } \frac{c}{d} \text{ or } \frac{e}{f} = \frac{a+c+e}{b+d+f}.$$

Hence, $$\frac{A}{B} \text{ or } \frac{C}{D} \text{ or } \frac{E}{F} = \frac{A+C+E}{B+D+F}.$$

Thus, **if any number of ratios be equal, where the antecedents and consequents are all magnitudes of the same kind, then any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.**

This inference is spoken of as **Addendo**.

18. A : B : : C : D, where A, B, C, D are four straight lines, then the rectangle contained by A and D is equal to the rectangle contained by B and C. For, if a, b, c, d be the numerical measures of A, B, C, D respectively, referred to one and the same unit, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \quad \frac{a}{b} \times bd = \frac{c}{d} \times bd,$$

or, $$ad = bc.$$

But ad and bc are the numerical measures of the areas of the rectangles A.D and B.C respectively.

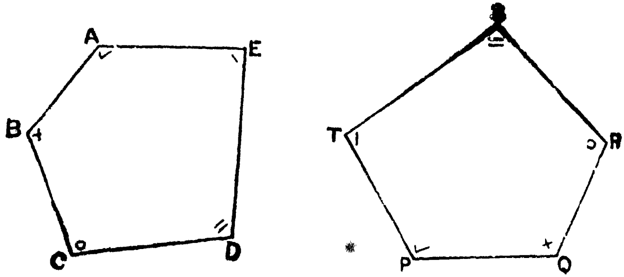
Hence the rectangle contained by A and D = the rectangle contained by B and C.

Thus, **if four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means.**

Conversely, if four straight lines A, B, C, D are such that A.D = B.C, then A : B = C : D.

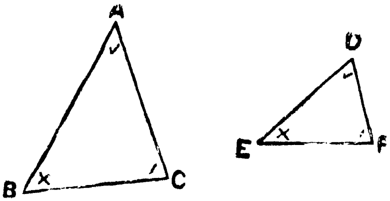
Note. It follows from the above that when three straight lines are in continued proportion the rectangle contained by the extremes is equal to the square on the mean. Conversely, if three straight lines A, B, C are such that $AC = B^2$, then A : B = B : C.

19. Two rectilineal figures are said to be **equiangular** when the angles of the one, taken in order, are respectively equal to those of the other, taken in order.



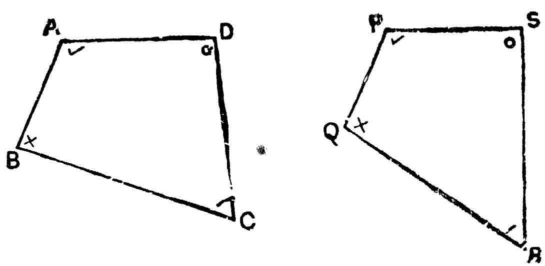
In the preceding diagram, the pentagons ABCDE and PQRST are *equiangular*, the ∠s A, B, C, D, E being respectively equal to the ∠s P, Q, R, S, T.

20. In two equiangular triangles, any angle of the first and that angle of the second to which it is equal are said to be **corresponding angles**; and sides *opposite* to corresponding angles are said to be **corresponding sides.** Thus, if in the $\triangle$s ABC, DEF, the $\angle$s A, B, C are respectively equal to the $\angle$s D, E, F, then



the $\angle$s A and D are a pair of *corresponding angles*; and so are the $\angle$s B and E, as well as the $\angle$s C and F. Also the sides BC and EF are a pair of *corresponding sides*; and so are the sides CA and FD, as well as the sides AB and DE.

Note. Generally speaking, in two *equiangular* rectilineal figures, any angle of the first and that angle of the second to which it is equal



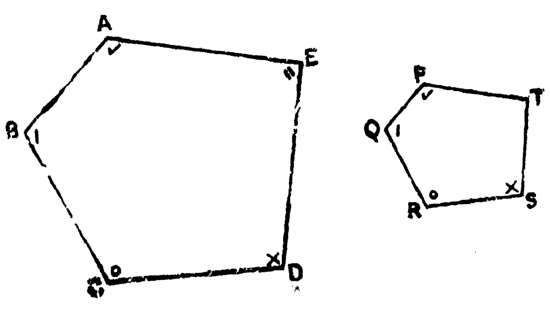
are said to be *corresponding angles*; also the common arm of any two consecutive angles of the first and the common arm of those two

consecutive angles of the second to which they are respectively equal are said to be **corresponding sides** Thus, if in the preceding two figures ABCD and PQRS, the ∠s A, B, C, D be respectively equal to the ∠s P, Q, R, S then the ∠s A and P are a pair of corresponding angles; so are the ∠s B and Q, the ∠s C and R, and the ∠s D and S. Again the ∠s B and C being respectively equal to the ∠s Q and R, the side BC (the common arm of the ∠s B and C) and the side QR (the common arm of the ∠s, Q and R) are a pair of *corresponding sides*; so are CD and RS, DA and SP, AB and PQ.

[Of two *corresponding angles*, or two *corresponding sides*, either is said to *correspond* to the other.]

21. If two rectilineal figures are equiangular and if the ratio of any side of the first to the corresponding side of the second is the same, then they are said to be **similar**.

Thus, in the following diagram, the figures ABCDE and PQRST are *similar*,

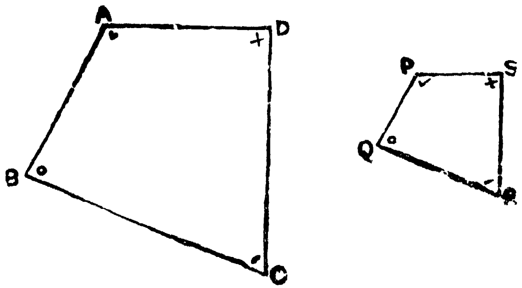


(i) if the ∠s A, B, C, D, E are respectively equal to the ∠s P, Q, R, S, T,

and (ii) if $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$.

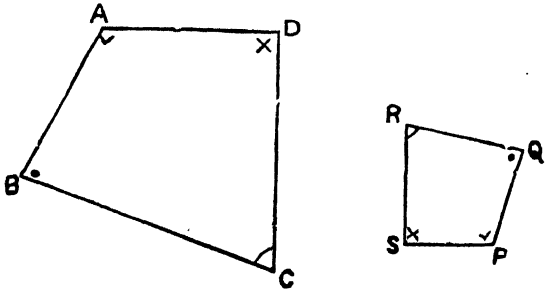
22. Two similar rectilineal figures are said to be **similarly situated** when their corresponding sides are parallel and the same in *sense*. Thus, in diagram (1), the two figures are

(1)



similar as well as *similarly situated*; whilst in diagram (2)

(2)

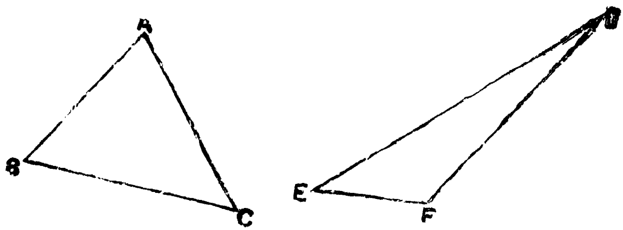


the two figures are similar but *not* similarly situated because the corresponding sides, though parallel, are opposite in *sense*.

23. Two rectilineal figures are said to have their sides about two of their angles **reciprocally proportional** when a side of the first is to a side of the second as the remaining side of the second is to the remaining

side of the first. Thus, the $\triangle$s ABC and DEF, have their sides about the $\angle$s B and E *reciprocally proportianal*,

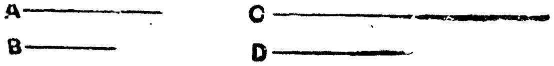
$$\text{if } \frac{AB}{DE} = \frac{EF}{BC}.$$



24. If four str. lines A, B, C, D be such that

$$\frac{A}{B} = \frac{C}{D}, \text{ then also}$$

$$\frac{\text{the sq. on A}}{\text{the sq. on B}} = \frac{\text{the sq. on C}}{\text{the sq. on D}}.$$



Let m, n be the numerical measures of A and B referred to some common unit ; then m and n are also the numerical measures of C and D referred to some *other* unit. The units of length being different, the units of area are also different in the two cases.

If α denote the unit of area in the first case and β, that in the second, we must have

$$\left.\begin{array}{l}\text{Sq. on A} = m^2\alpha \\ \text{sq. on B} = n^2\alpha\end{array}\right\} \qquad \left.\begin{array}{l}\text{sq. on C} = m^2\beta \\ \text{sq. on D} = n^2\beta\end{array}\right\}.$$

Hence $\dfrac{\text{sq. on A}}{\text{sq. on B}} = \dfrac{\text{sq. on C}}{\text{sq. on D}}$, (each of these two ratios being $= \dfrac{m^2}{n^2}$).

EXERCISE (28).

1. Define the *unit of length* and *unit of area*. If 3 inches be the unit of length, what is the unit of area? Into how many units of area can a rectangle be divided whose sides are 9 and 15 inches respectively? Illustrate your answer by a diagram.

2. When is one magnitude said to be a *multiple* of another? and when a *measure*? If two magnitudes have a common multiple, prove that they have also a common measure.

3. When are two magnitudes said to be *commensurable*? If A and B denote any two magnitudes that are commensurable, prove that, A being divided into a certain number of equal parts, any of these parts is contained an exact number of times in B.

4. When are two magnitudes of the same kind said to be *incommensurable*? If the length of the diagonal of a square be d inches, show that a side of the square = ($d \times \cdot 70710678\ldots$) inches. Hence prove that the diagonal and the side have no common measure.

5. In the preceding example, shew that the diagonal and the side may be regarded as having a common measure, if a very small error be neglected.

6. What is meant by the *ratio* of one magnitude to another of the same kind? Give illustrations.

7. What is the meaning of "$\dfrac{A}{B} = \dfrac{C}{D}$"?

For the purpose of this relation, is it necessary that A, B, C, D should all be magnitudes of the same kind? If not, how may the matter stand?

8. If three magnitudes A, B, C of the same kind be such that $\frac{B}{A} = \frac{C}{A}$, prove that B = C.

9. If four magnitudes A, B, C, D be such that $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{B}{A} = \frac{D}{C}$.

10. If four magnitudes A, B, C, D of the same kind be such that $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A}{C} = \frac{B}{D}$.

11. If A, B, C, D be four straight lines such that $\frac{A}{B} = \frac{C}{D}$, prove that the rect. A, D = the rect. B, C.

12. If A, B, C be three given straight lines such that $A.C = B^2$, prove that $\frac{A}{B} = \frac{B}{C}$.

13. When are two rectilineal figures said to be *equiangular*? Illustrate your meaning by a diagram.

14. What are *corresponding angles* and *corresponding sides* in two equiangular rectilineal figures? Illustrate your meaning by a diagram.

15. When are two rectilineal figures said to be *similar*? Illustrate your meaning by a diagram.

16. When are two similar rectilineal figures said to be *similarly situated*? Illustrate your meaning by a diagram.

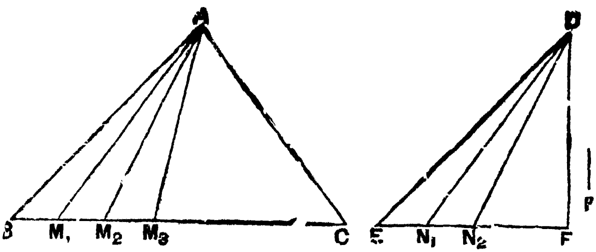
17. If four straight lines A, B, C, D be such that $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A^2}{B^2} = \frac{C^2}{D^2}$.

SECTION II.

THEOREMS.

Theorem 1. (EUC. VI. 1.)

If two triangles have equal altitudes their areas are to one another as their bases.



Let the $\triangle$s ABC, DEF, standing on bases BC and EF respectively, have equal altitudes.

To prove that the $\triangle$ABC : the $\triangle$DEE
= BC : EF.

Proof. Let the str. line P be a common measure of BC and EF; and suppose that P is contained m times in BC and n times in EF. Then $\dfrac{BC}{EF} = \dfrac{m}{n}$. . . . . . . , (1)

From BC cut off BM_1, M_1M_2, M_2M_3, &c., each = P; and from EF cut off EN_1, N_1N_2, &c., each = P. Then BC is divided into m *equal* parts at M_1, M_2, M_3, &c., and EE is divided into n *equal* parts at N_1, N_2, &c.

Join AM_1, AM_2, AM_3, &c., DN_1, DN_2, &c.

Now, it is clear that $BM_1 = M_1M_2 = M_2M_3 =$ &c. $= EN_1 = N_1N_2 =$ &c., each of them being $= P$.

Hence, the $\triangle ABC = m$ times the $\triangle ABM_1$;
and the $\triangle DEF = n$ times the $\triangle DEN_1$.

But the $\triangle ABM_1 =$ the $\triangle DEN_1$, because they stand on equal bases and have equal altitudes.

[*Bk. II, Th. 2, Note 1.*]

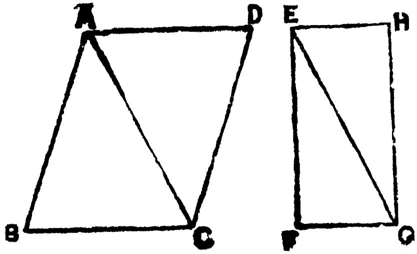
Hence, if a denote the area of either of these two $\triangle$s, we have

$$\left.\begin{array}{ll} & \text{the } \triangle ABC = ma \\ \text{and} & \text{the } \triangle DEF = na \end{array}\right\}$$

Hence, $$\frac{\text{the } \triangle ABC}{\text{the } \triangle DEF} = \frac{m}{n}.$$

and $\therefore$ from (1), $$\frac{\text{the } \triangle ABC}{\text{the } \triangle DEE} = \frac{BC}{EF};$$

Cor. *If two parallelograms have equal altitudes their areas are to one another as their bases.*



Let the par^{ms}. AC and EG, standing on the bases BC and FG respectively, have equal altitudes. Join AC, EG. Then the $\triangle$s ABC, EFG have also equal altitudes.

$$\left.\begin{array}{ll}\text{Now, the par}^{m}.\ \mathrm{AC} : \text{the } \triangle\mathrm{ABC} = 2 \\ \text{and the par}^{m}.\ \mathrm{EG} : \text{the } \triangle\mathrm{EFG} = 2\end{array}\right\}$$

$$\therefore \quad \frac{\text{the par}^{m}.\ \mathrm{AC}}{\text{the } \triangle\mathrm{ABC}} = \frac{\text{the par}^{m}.\ \mathrm{EG}}{\text{the } \triangle\ \mathrm{EFG}};$$

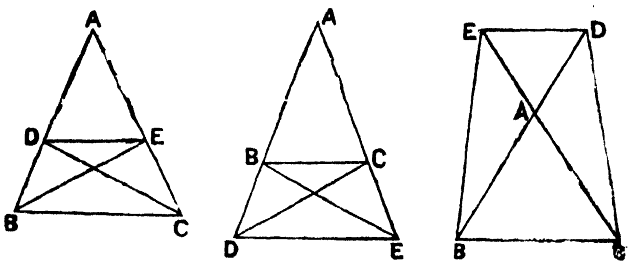
$$\therefore, \textit{alternately}, \frac{\text{the par}^{m}.\ \mathrm{AC}}{\text{the par}^{m}.\ \mathrm{EG}} = \frac{\text{the } \triangle\mathrm{ABC}}{\text{the } \triangle\mathrm{EFG}} = \frac{\mathrm{BC}}{\mathrm{FG}}.$$

Note. In the first diagram, if BC and EF be *incommensurable* either of them may be regarded as a *very large* multiple of a *very small* measure of the other (*Sec. I, Art.* 8.). Hence the str. line P will be indefinitely small, and consequently m and n will be indefinitely large.

Theorem 2. (Euc. VI. 2.)

If a straight line is drawn parallel to one side of a triangle, it cuts the other two sides, or those sides produced, proportionally.

Conversely, if a straight line cuts two sides of a triangle, or the two sides produced, proportionally, then it is parallel to the remaining sides of the triangle.



(i) Let DE be drawn ∥ to the side BC of the △ABC, cutting AB, AC, or those sides produced, at D and E.

To prove that AD : DB = AE : EC.

Proof. Join BE, CD.

Now, the △s BDE, CED are on the same base DE and between the same ∥s DE, BC;

∴ the △ BDE = the △ CED.

∴ the △ ADE : the △ BDE = the △ ADE : the △ CED.

$$\left.\begin{aligned}\text{But}\quad \frac{\text{the } \triangle \text{ ADE}}{\text{the } \triangle \text{ BDE}} &= \frac{\text{AD}}{\text{DB}}\\ \text{and}\quad \frac{\text{the } \triangle \text{ ADE}}{\text{the } \triangle \text{ CED}} &= \frac{\text{AE}}{\text{EC}}\end{aligned}\right\};\qquad (Th.\ 1.)$$

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC}. \qquad \text{Q. E. D.}$$

(ii) Let the str. line DE cut the sides AB, AC, or those sides produced, of the $\triangle$ ABC, proportionally; so that AD : DB = AE : EC.

To prove that DE is $\parallel$ to BC.

Proof. Join BE, CD.

$$\left.\begin{aligned} \text{Now,} \quad \frac{AD}{DB} &= \frac{\text{the } \triangle \text{ ADE}}{\text{the } \triangle \text{ BDE}} \\ \text{and} \quad \frac{AE}{EC} &= \frac{\text{the } \triangle \text{ ADE}}{\text{the } \triangle \text{ CED}} \end{aligned}\right\}. \qquad (\textit{Th. 1.})$$

$$\text{But} \quad \frac{AD}{DB} = \frac{AE}{EC}; \qquad (\textit{Hyp.})$$

$$\therefore \quad \frac{\text{the } \triangle \text{ ADE}}{\text{the } \triangle \text{ BDE}} = \frac{\text{the } \triangle \text{ ADE}}{\text{the } \triangle \text{ CED}};$$

$$\therefore \quad \text{the } \triangle \text{ BDE} = \triangle \text{ CED}.$$

But these two $\triangle$s are on the same base DE and on the same side of it;

$\therefore$ DE is $\parallel$ to BC. (*Bk. II, Th. 4, Cor.*) Q. E. D.

Cor. 1. *If a straight line drawn parallel to the side* BC *of a triangle* ABC *cuts the sides* AB, AC *in* D *and* E, *then*

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

For, in fig. (1), $\frac{AD}{DB} = \frac{AE}{EC}$,

$\therefore$ inversely, $\frac{DB}{AD} = \frac{EC}{AE}$;

$$\therefore \quad \frac{DB + AD}{AD} = \frac{EC + AE}{AE} \quad \text{(Componendo)},$$

i.e., $$\frac{AB}{AD}=\frac{AC}{AE}.$$

Hence, inversely, $$\frac{AD}{AB}=\frac{AE}{AC}.$$

Cor. 2. *If points* **D** *and* **E** *be on the sides* **AB**, **AC** *of a triangle* **ABC**, *so that* $\frac{AD}{AB}=\frac{AE}{AC}$, *then* **DE** *is parallel to* **BC.**

Referring to fig. (1), since $\frac{AD}{AB}=\frac{AE}{AC}$,

$\therefore$ inversely, $$\frac{AB}{AD}=\frac{AC}{AE};$$

$\therefore$ $$\frac{AB-AD}{AD}=\frac{AC-AE}{AE} \text{ (Dividendo)},$$

i.e., $$\frac{DB}{AD}=\frac{EC}{AE}.$$

Hence, inversely, $$\frac{AD}{DB}=\frac{AE}{EC},$$

and $\therefore$ DE is $\parallel$ to BC.

EXERCISE (29).

1. Prove that the rectangle contained by two lines is a mean proportional between their squares.

2. If two triangles stand on the same base, prove that their areas are to one another as their altitudes.

3. If the medians BE, CF of a given triangle ABC intersect at G, prove that the distances of the points B and C from AG are equal, and hence shew that AG produced bisects BC.

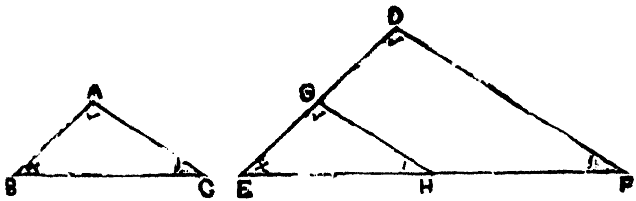
4. ABCD is a trapezium of which AB, DC are the parallel sides. If any straight line EF parallel to AB cuts AD, BC in E and F respectively, prove that AE : ED = BF : FC.

5. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD meeting BC, BD in F and G respectively. Prove that FG is parallel to CD.

6. ABC is a triangle; and DE is drawn parallel to BC, meeting AB, AC in D and E respectively. If BE, CD intersect at F, prove that the triangle ADF is equal to the triangle AEF.

Theorem 8. (Euc. VI. 4.)

If two triangles are equiangular, their corresponding sides are proportional.



Let the $\triangle$s ABC, DEF be equiangular, having the $\angle$s A, B, C respectively equal to the $\angle$s D, E, F.

To prove that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Proof. Apply the $\triangle$ ABC to the $\triangle$ DEF so that B falls on E and BA on ED, BC falling on the same side of ED as EF.

Then, since the $\angle$ B = the $\angle$ E, BC *must* fall on EF.

Let the pt. G on ED be the new position of A, and the pt. H on EF the new position of C; then GEH is the new position of the $\triangle$ABC.

Hence, the $\angle$EGH = the $\angle$EDF, and $\therefore$ GH is $\parallel$ to DF.

Hence, $\frac{EG}{ED} = \frac{EH}{EF}$; *(Th. 2, Cor. 1.)*

i.e. $\frac{AB}{DE} = \frac{BC}{EF}$.

Similarly, by applying the $\triangle ABC$ to the $\triangle DEF$ so that the $\angle C$ coincides with the $\angle F$, it may be proved that

$$\frac{BC}{EF} = \frac{CA}{FD}.$$

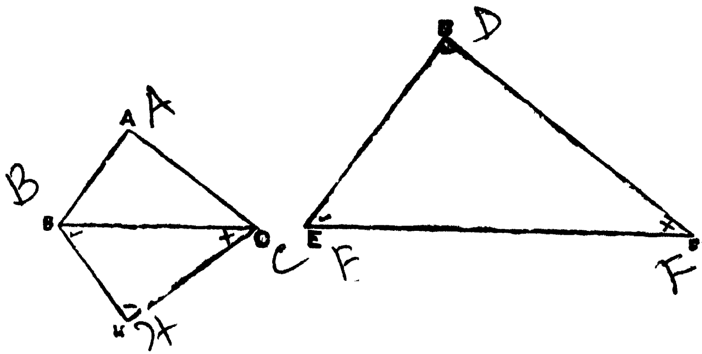
Thus, we have $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$. Q. E. D.

Note. If two triangles are equiangular and also have their *corresponding* sides proportional, then they are said to be **similar**. It is proved that if the first condition be satisfied, the second must *necessarily* follow. Hence, two triangles that are equiangular are necessarily *similar*.

Cor. *In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.*

Theorem 4. (Euc. VI. 5.)

If two triangles have their sides proportional, then they are also equiangular, having those angles equal which are opposite to corresponding sides.



Let the $\triangle$s ABC, DFF be such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

To prove that the $\angle C$ = **the** $\angle F$, **the** $\angle A$ = **the** $\angle D$, **and the** $\angle B$ = **the** $\angle E$.

Proof. Make the $\angle CBH$ = the $\angle E$, and the $\angle BCH$ = the $\angle F$; then the $\angle BHC$ = the $\angle D$.

Now the $\triangle$s HBC, DEF are equiangular ;

$$\therefore \qquad \frac{HB}{DE} = \frac{BC}{EF}.$$

But, by hypothesis,

$$\frac{AB}{DE} = \frac{BC}{EF} \; ; \; \therefore \frac{HB}{DE} = \frac{AB}{DE},$$

$$\left.\begin{array}{r} \text{and } \therefore \; HB = AB. \\ \text{Similarly, } HC = AC. \end{array}\right\}$$

Now in the $\triangle$s ABC, HBC, we have

(1) $AB = HB$.

(2) $AC = HC$.

and (3) BC common;

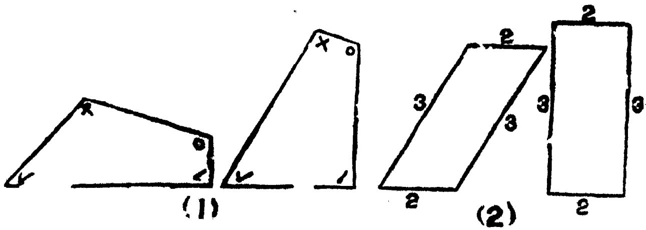
$\therefore$ the two $\triangle$s are congruent.

$$\left.\begin{aligned} \text{Hence, the } \angle ACB &= \text{the } \angle HCB = \text{the } \angle F\,; \\ \text{the } \angle BAC &= \text{the } \angle BHC = \text{the } \angle D\,; \\ \text{and the } \angle ABC &= \text{the } \angle HBC = \text{the } \angle E. \end{aligned}\right\}$$

Q. E. D.

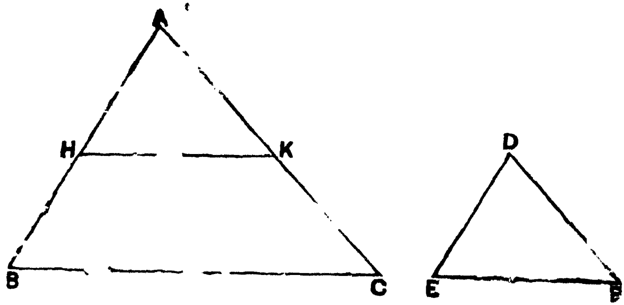
Note 1. From this proposition and the last, it is clear that of the two conditions that must be satisfied in order that two triangles may be *similar*, each necessarily follows from the other. Hence for the similarity of two triangles, it is sufficient that *either* they should be equiangular, *or* they should have their sides proportional.

Note 2. In the case of rectilineal figures other than triangles, however either of the two conditions of similarity does *not* necessarily follow from the other. For instance, in the first of the following diagrams, the two rectilineal figures are equiangular, but their corresponding sides are not proportional; whereas in the second, the two figures have their sides proportional, but they are not equiangular.



Theorem 5. (Euc. VI 6.)

If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, then the triangles are similar.



Let the $\triangle$s ABC, DEF have the $\angle$ A = the $\angle$ D, and let $\frac{AB}{DE}$ be = $\frac{AC}{DF}$

To prove that the two $\triangle$s are similar.

Proof. Apply the $\triangle$DEE to the $\triangle$ABC so that D falls on A and DE on AB, DF falling on the same side of AB as AC.

Then, since the $\angle$ A = the $\angle$ D, DF must fall on AC.

Let the pt. H on AB and the pt. K on AC be the new positions of E and F respectively; then AHK is the new position of the $\triangle$DEF.

$$\text{Now, } \frac{AB}{DE} = \frac{AC}{DF}; \qquad (Hyp.)$$

$$i.e., \quad \frac{AB}{AH} = \frac{AC}{AK}.$$

$\therefore$ HK is $\parallel$ to BC. (*Th. 2, Cor. 2*),

Hence the $\angle$ AHK (which is the $\angle$ E) = the $\angle$ B; and the $\angle$ AKH (which is the $\angle$ F) = the $\angle$ C.

Thus, the two $\triangle$s ABC, DEF are equiangular;

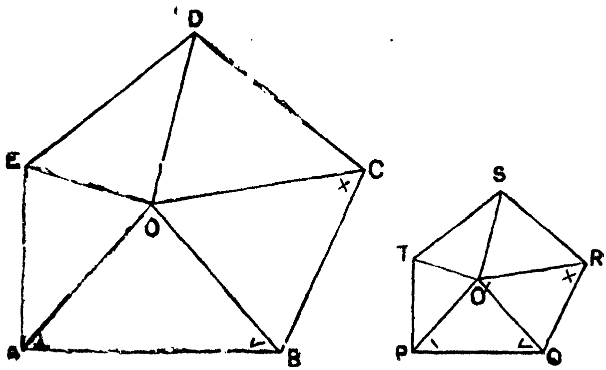
and $\therefore$ their corresponding sides are proportional.

(*Th.* 3.)

Hence, the $\triangle$s ABC, DEF are similar. Q. E. D

Theorem 6.

If a polygon is divided into triangles by lines joining a given point to its vertices, any similar polygon can be divided into corresponding similar triangles.



Let ABCDE, PQRST be two similar polygons, having the $\angle$s A, B, C, D, E respectively equal to the $\angle$s P, Q, R, S, T; and having

$$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{CD}{RS}=\frac{DE}{ST}=\frac{EA}{TP}.$$

Let the polygon ABCDE be divided into the $\triangle$s AOB, BOC, COD, DOE, EOA by joining any pt. O within the polygon to its vertices.

To prove that there is a point O′ within the polygon PQRST such that the $\triangle$PO′Q is similar to the $\triangle$AOB, the $\triangle$QO′R is similar to the $\triangle$BOC, and so on.

Proof. Make the $\angle$QPO′ = the $\angle$BAO, and the $\angle$PQO′ = the $\angle$ABO.

Join O′R, O′S, O′T.

(1) The $\triangle$s AOB, PO′Q are equiangular;

$\therefore$ they are similar. (*Th.* 3, *Note.*)

(2) The $\angle$ ABC = the $\angle$ PQR, (*Hyp.*)

and the $\angle$ ABO = the $\angle$ PQO′;

$\therefore$ the $\angle$ OBC = the $\angle$ O′QR. ... ... (α)

Also, the $\triangle$s AOB, PO′Q being similar,

$$\frac{OB}{O'Q} = \frac{AB}{PQ};$$

but $$\frac{AB}{PQ} = \frac{BC}{QR};$$ (*Hyp.*)

$\therefore$ $$\frac{OB}{O'Q} = \frac{BC}{QR}.$$ ... ... (β)

Hence, from (α) and (β),

the $\triangle$s BOC, QO′R are similar. (*Th.* 5.)

(3) The $\angle$ BCD = the $\angle$ QRS, (*Hyp.*)

and the $\angle$ BCO = the $\angle$ QRO′; ($\triangle$sBOC, QO′R being equiangular.

$\therefore$ the $\angle$ OCD = the $\angle$ O′RS ... ... (γ)

Also, the $\triangle$s BOC, QO′R being similar,

$$\frac{OC}{O'R} = \frac{BC}{QR};$$

but $$\frac{BC}{QR} = \frac{CD}{RS};$$ (*Hyp.*)

$\therefore$ $$\frac{OC}{O'R} = \frac{CD}{RS}$$ ... ... (δ)

Hence, from (γ) and (δ),

the $\triangle$s COD, RO′S are similar. (*Th.* 5.)

In the same way,

the $\triangle$s DOE, SO′T are similar;

and so are the $\triangle$s EOA, TO′P. Q. E. D.

Note. If the pt. **O** coincides with one of the angular points of the figure **ABCDE**, then **O′** will coincide with the corresponding angular point of the figure **PQRST**. For instance, if **O** coincides with **A**, **O′** will coincide with **P**.

EXERCISE (30).

1. ABC is a triangle, and DE is drawn parallel to BC, meeting AB, AC in D and E respectively. If BE, CD intersect in O, prove that the area of the triangle COE is a mean proportional between the areas of the triangles DOE and BOC.

2. B is the mid-point of a straight line AC, and the triangles ADB, BEC are between the same parallels. If a straight line parallel to BC cuts DA, DB, EB, EC in F, G, H, K respectively, prove that FG = HK.

3. If the numerical measures of the sides of one triangle be 2, 3, 4, and those of another, 6, 9, 12, prove that the two triangles are equiangular.

4. ABC is a triangle; BE, CF are drawn to the opposite sides so as to intersect at G on the median from A. Prove that the $\triangle$AGC : the $\triangle$BGC = AF : FB, and hence prove that FE is parallel to BC.

5. AB and CD are two parallel straight lines, and E is the mid-point of CD; AC and BE meet at F, and AE and BD meet at G. Prove that FG is parallel to AB.

6. Two straight lines AB, CD intersect at B, so that OA : OC = OD : OB. Prove that the ∠ADO = the ∠CBO, and hence shew that the four points A, B, C, D are concyclic.

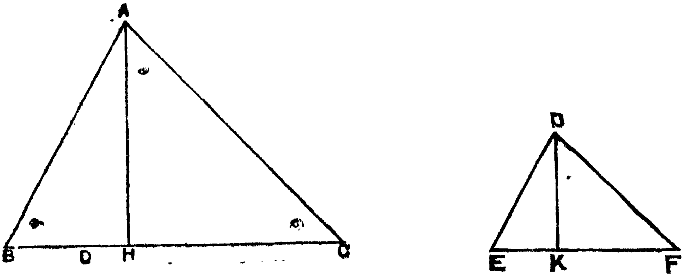
7. In a △ABC, AD is drawn perpendicular to BC. If AD is the mean proportional to BD and DC, prove that the ∠BAC is a rt. ∠.

8. AC and BD are drawn perpendiculars to a given straight line CD from two given points A and B ; AD and BC intersect in E, and EF is drawn perpendicular to CD. Prove that AF and BF are equally inclined to CD.

9. If a polygon is divided into triangles having one of the vertices of the polygon for a common vertex, shew that any similar polygon can be divided into corresponding similar triangles.

Theorem 7. (Euc. VI. 19.)

The ratio of the areas of two similar triangles is equal to the ratio of the squares on the corresponding sides.



Let ABC, DEF be two similar $\triangle$s having the $\angle$s A, B, C, respectively equal to the $\angle$s D, E, F; so that BC and EF are a pair of corresponding sides.

To prove that

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}.$$

Proof. Let AH, DK be $\perp$s to BC, EF; and let a, d, h, k be the numerical measures of BC, EF, AH, DK respectively, referred to one and the same unit.

$$\left.\begin{array}{l}\text{Then the area of the } \triangle ABC = \tfrac{1}{2}ah \text{ units of area;}\\ \text{and the area of the } \triangle DEF = \tfrac{1}{2}dk \text{ units of area.}\end{array}\right\}$$

$$\therefore \quad \frac{\triangle ABC}{\triangle DEF} = \frac{ah}{dk} = \frac{a}{d} \cdot \frac{h}{k}.$$

$$\left.\begin{array}{l}\text{Now, the rt. } \angle B = \text{the } \angle E,\\ \text{and the rt. } \angle AHB = \text{the rt. } \angle DKE;\end{array}\right\}$$

$\therefore$ the $\triangle$s AHB, DKE are equiangular.

Hence $$\frac{AH}{DK} = \frac{AB}{DE};$$

but $$\frac{AB}{DE} = \frac{BC}{EF}; \qquad (Hyp).$$

$$\therefore \quad \frac{AH}{DK} = \frac{BC}{EF}, \quad \text{and} \quad \therefore \quad \frac{h'}{k} = \frac{a}{d}.$$

Hence, $$\frac{\triangle ABC}{\triangle DEF} = \frac{a}{d} \cdot \frac{a}{d} = \frac{a^2}{d^2}.$$

But a^2 = number of units of area in the square on BC,
and d^2 = number of units of area in the square on EF,

$$\therefore \quad \frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}. \qquad \text{Q. E. D.}$$

Note 1. Let a, b, c, d, e, f, be the numerical measures of **BC**, **CA**, **AB**, **EF**, **FD**, **DE**, respectively referred to one and the same unit.

Then, since $\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}$ we must have $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$;

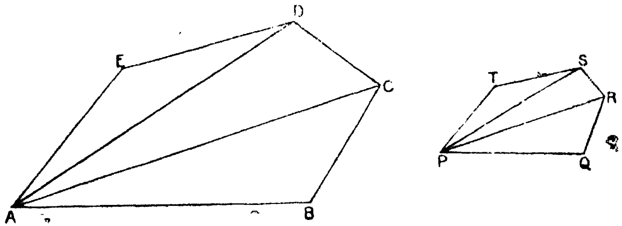
and $\therefore \frac{a^2}{d^2} = \frac{b^2}{e^2} = \frac{c^2}{f^2}$. Hence, $\frac{BC^2}{EF^2} = \frac{CA^2}{FD^2} = \frac{AB^2}{DE^2}$.

Hence $\frac{\triangle ABC}{\triangle DEF} = \frac{CA^2}{FD^2}$ and also $= \frac{AB^2}{DE^2}$.

Note 2. That $\frac{\triangle ABC}{\triangle DEF} = \frac{CA^2}{FD^2}$ may also be proved independently by drawing perpendiculars from **B** and **E** on **AC** and **DF** respectively.

Theorem 8. (Euc. VI. 20.)

The ratio of the areas of two similar polygons is equal to the ratio of the squares on the corresponding sides.



Let ABCDE, PQRST, be two similar polygons, having the $\angle$s A, B, C, D, E respectively equal to the $\angle$s P, Q, R, S, T; so that AB and PQ are a pair of corresponding sides.

To prove that

$$\frac{\text{fig. ABCDE}}{\text{fig. PQRST}} = \frac{AB^2}{PQ^2}.$$

Proof. Join AC, AD, PR, PS.

$$\left.\begin{array}{ll} \text{Then} & \text{the } \triangle\text{s ABC, PQR are similar,} \\ & \text{the } \triangle\text{s ACD, PRS are similar;} \\ \text{and} & \text{the } \triangle\text{s ADE, PST are similar.} \end{array}\right\} \qquad (Th.\ 6.)$$

$$\left.\begin{array}{ll} \text{Hence} & \dfrac{\triangle ABC}{\triangle PQR} = \dfrac{AC^2}{PR^2}; \\ \text{also} & \dfrac{\triangle ACD}{\triangle PRS} = \dfrac{AC^2}{PR^2}; \end{array}\right\} \qquad (Th.\ 7.)$$

$$\therefore \quad \frac{\triangle ABC}{\triangle PQR} = \frac{\triangle ACD}{\triangle PRS}. \qquad \dots \dots \dots (1)$$

$$\text{Similarly,} \quad \frac{\triangle ACD}{\triangle PRS} = \frac{\triangle ADE}{\triangle PST} \qquad \dots \dots \dots (2)$$

Hence, from (1) and (2),

$$\frac{\triangle ABC}{\triangle PQR} = \frac{\triangle ACD}{\triangle PRS} = \frac{\triangle ADE}{\triangle PST}.$$

Therefore each of these ratios

$$= \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle PQR + \triangle PRS + \triangle PST}, \qquad (Addendo)$$

$$= \frac{\text{fig. } ABCDE}{\text{fig. } PQRST}.$$

Hence, $\dfrac{\text{fig. } ABCDE}{\text{fig. } PQRST} = \dfrac{\triangle ABC}{\triangle PQR} = \dfrac{AB^2}{PQ^2}$. Q. E. D.

EXERCISE (31).

1. ABC is a triangle, and D is the mid-pt. of the semicircle described on AB as diameter; E is a point on AB so that $AE = AD$. Prove that the straight line drawn through E parallel to BC bisects the triangle.

2. Shew how to draw a line parallel to the base of a triangle so as to form with the other two sides produced a triangle double of the given triangle.

3. ABC is a right-angled triangle of which BC is the hypotenuse; AD is drawn perpendicular to BC. Prove that $BD : DC = AB^2 : AC^2$.

4. ABC is a triangle inscribed in a circle, and the tangent at A meets BC produced in D. Prove that

$$BD : CD = AB^2 : AC^2.$$

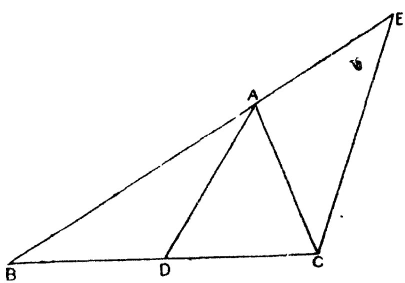
5. BE, CF are the perpendiculars on the sides CA, AB of a triangle ABC. Prove that the $\triangle ABC$: the $\triangle AFE$ $= AB^2 : AE^2$.

6. If two regular pentagons be inscribed in two given circles, prove that their areas are to one another as the squares on the radii of the circles.

Theorem 9. (Euc. VI. 3.)

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

Conversely, if one side of a triangle is divided internally in the ratio of the other two sides, then the line drawn from the point of division to the opposite vertex is the internal bisector of the angle at that vertex.



(i) In the $\triangle ABC$, let AD be the internal bisector of the $\angle BAC$, meeting BC in D.

To prove that $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Proof. Let CE be $\parallel$ to DA, meeting BA produced at E.

Then the $\angle AEC$ = the ext. $\angle BAD$;
and the $\angle ACE$ = the alt. $\angle CAD$.

But the $\angle BAD$ = the $\angle CAD$; (*Hyp.*)

$\therefore$ the $\angle AEC$ = the $\angle ACE$.

$\therefore$ AC = AE ... ... (*a*)

Now, in the $\triangle$BCE, DA is $\parallel$ to CE ;

$$\therefore \qquad \frac{BD}{DC} = \frac{BA}{AE}. \qquad (Th.\ 2.)$$

Hence, from (α), $\dfrac{BD}{DC} = \dfrac{BA}{AC}$. Q. E. D.

(ii) In the $\triangle$ABC, let the side BC be divided internally at D so that $\dfrac{BD}{DC} = \dfrac{BA}{AC}$; and join AD.

To prove that the $\angle$BAD = the $\angle$CAD.

Proof. Let CE be $\parallel$ to DA, meeting BA produced in E.

Then in the $\triangle$BCE, DA is $\parallel$ to CE ;

$$\therefore \qquad \frac{BD}{DC} = \frac{BA}{AE} \qquad (Th.\ 2.)$$

$$\text{But} \qquad \frac{BD}{DC} = \frac{BA}{AC}\ ; \qquad (Hyp.)$$

$$\therefore \qquad \frac{BA}{AC} = \frac{BA}{AE}\text{, and } \therefore\ AC = AE.$$

Hence the $\angle$AEC = the $\angle$ACE. ... ... (β)

Now, since DA is $\parallel$ to CE,

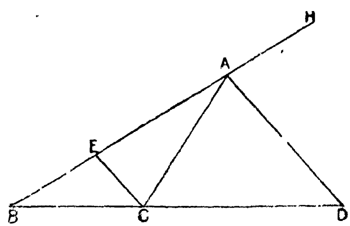
$$\left.\begin{array}{l}\therefore \quad \text{the } \angle AEC = \text{the ext. } \angle BAD\ ; \\ \text{and the } \angle ACE = \text{the alt } \angle CAD.\end{array}\right\}$$

Hence, from (β),

the $\angle$BAD = the $\angle$CAD. Q. E. D.

Theorem 10. (Euc. VI. A.)

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle. Conversely, if one side of a triangle is divided externally in the ratio of the other two sides, then the line drawn from the point of division to the opposite vertex is the external bisector of the angle at that vertex.



(i) In the $\triangle ABC$, let H be any point on BA produced and let AD be the bisector of the $\angle HAC$, meeting BC produced in D; then AD is the *external* bisector of the $\angle BAC$.

To prove that $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Proof. Let CE be $\parallel$ to DA, meeting BA in E.

Then the $\angle AEC$ = the ext. $\angle HAD$;
and the $\angle ACE$ = the alt. $\angle CAD$.

But the $\angle HAD$ = the $\angle CAD$, (*Hyp.*)

$\therefore$ the $\angle AEC$ = the $\angle ACE$;

$\therefore$ AC = AE. ... ... (a)

Now, in the $\triangle BCE$, DA is $\parallel$ to CE;

$$\therefore \quad \frac{BD}{DC} = \frac{BA}{AE}. \qquad (Th.\ 2.)$$

Hence, from (α), $\dfrac{BD}{DC} = \dfrac{BA}{AC}$. Q. E. D.

(ii) In the $\triangle ABC$, let the side BC be divided externally at D so that $\dfrac{BD}{DC} = \dfrac{BA}{AC}$.

Join AD and produce BA to any point H.

To prove that AD **is the bisector of the** $\angle HAC$.

Proof. Let CE be $\parallel$ to DA, meeting BA in E.

Then in the $\triangle BCE$, DA is $\parallel$ to CE;

$$\therefore \quad \frac{BD}{DC} = \frac{BA}{AE}. \qquad (Th. 2.)$$

$$\text{But} \quad \frac{BD}{DC} = \frac{BA}{AC}; \qquad (Hyp.)$$

$$\therefore \quad \frac{BA}{AC} = \frac{BA}{AE}, \text{ and } \therefore \quad AC = AE.$$

Hence the $\angle AEC$ = the $\angle ACE$. ... ... (β)

Now, since DA is $\parallel$ to CE,

$$\left.\begin{array}{l} \therefore \text{ the } \angle AEC = \text{the ext. } \angle HAD; \\ \text{and the } \angle ACE = \text{the alt. } \angle CAD. \end{array}\right\}$$

Hence, from (β),

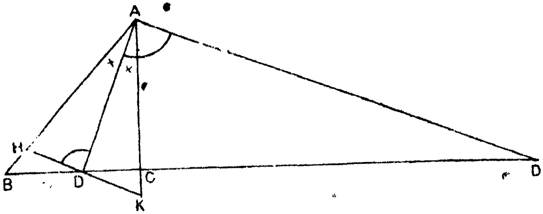
the $\angle HAD$ = the $\angle CAD$;

i.e., AD is the bisector of the $\angle HAC$. Q. E. D.

Cor. *If the side* **BC** *of a triangle* **ABC** *be divided internally at* **D** *and externally at* **D′** *in the ratio of* **BA** *to* **AC** *then* **DD′** *subtends a right angle at* **A**.

Note 1. If a straight line **BC** is divided internally at **D** and externally at **D′** in the same ratio (so that **BD : DC = BD′ : D′C**),

and if **DD′** subtends a right angle at any point **A**, then it may be proved that **AD** is the internal bisector of the angle **BAC.**



Let **HDK** be drawn ‖ to **AD′**, meeting **AB** in **H** and **AC** produced in **K**. Since **HK** is ‖ to **AD′**, ∴ the ∠**ADH** = the ∠**DAD′** = a rt.∠.

Since $\frac{BD}{DC} = \frac{BD'}{D'C}$, ∴, alternately $\frac{BD}{BD'} = \frac{DC}{D'C}$.

Now the △s **BDH**, **BD′A** being equiangular, we have $\frac{BD}{BD'} = \frac{DH}{D'A}$; and the △s **DCK**, **D′CA** being equiangular, we have $\frac{DC}{D'C} = \frac{DK}{D'A}$. Hence $\frac{DH}{D'A} = \frac{DK}{D'A}$, and ∴ **DH = DK**. Now it is easy to see that the △s **ADH**, **ADK** are congruent; and ∴ the ∠**BAD** = the ∠**CAD.**

Note 2. When a given straight line is divided internally and externally in the same ratio, it is said to be **cut harmonically.** Hence any side of a triangle may be said to be *cut harmonically* by the internal and external bisectors of the opposite vertical angle.

EXERCISE (32).

1. ABCD is a quadrilateral, and the bisectors of the angles A and C meet on the diagonal BD. Prove that the bisectors of the angles B and D meet on AC.

2. In a given circle, AB is a diameter and CD is a chord perpendicular to it ; through any point E in CD, AE and BE are drawn to meet the circle in F and G. Prov that the quadrilateral CFDG has any two of its adjacent sides in the same ratio as the other two.

3. ABC is a triangle, and BO, CO are the bisectors of the angles B and C. If AO produced meets BC in D, prove that AB : AC = BD : CD.

4. ABC is a triangle, and BP, CP are the bisectors of the exterior angles at B and C. If AP cuts BC at D, prove that AB : AC = BD : DC.

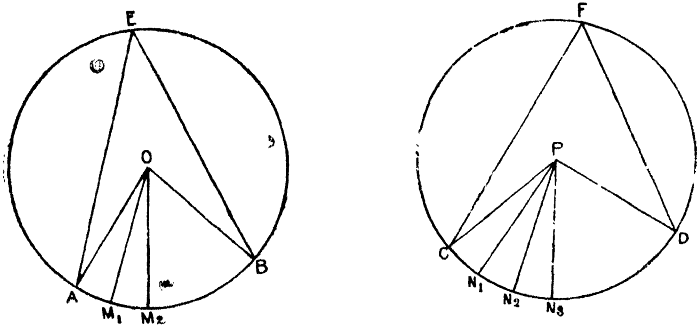
5. In a given circle, AB is a diameter, and P is any point on the circumference ; PC, PD are two straight lines on opposite sides of PA and equally inclined to it, meeting AB in C and D. Prove that AC : CB = AD : DB.

6. ABC is a triangle, and AD is the bisector of the angle A, meeting BC in D. Prove that AB : BD = AB + AC : BC.

Hence, if O be the in-centre of the triangle ABC, prove that AO : OD = AB + AC : BC.

Theorem 11. (Euc. VI. 33.)

In equal circles, angles, whether at the centres or at the circumferences, have the same ratio as the arcs on which they stand.



Let ABE, CDF be two equal ⊙s. of which the centres are O and P.

(i) the ∠s AOB, CPD at the centres stand on the arcs AB, CD.

To prove that $\dfrac{\text{the } \angle \text{ AOB}}{\text{the } \angle \text{ CPD}} = \dfrac{\text{the arc AB}}{\text{the arc CD}}$.

Proof. Suppose that the mth part of the arc AB is contained n times in the arc CD; *i.e.*, suppose that a common measure of the two arcs is contained m times in the arc AB, and n times in the arc CD.

Hence $\dfrac{\text{the arc AB}}{\text{the arc CD}} = \dfrac{m}{n}$.

Let the arc AB be divided into m equal parts at the pts. M_1, M_2, &c.; and let the arc CD be divided into n equal parts at N_1, N_2, N_3, &c.

Join OM_1, OM_2, &c., PN_1, PN_2, PN_3, &c,

Then the arcs AM_1, M_1M_2, &c., CN_1, N_1N_2, N_2N_3, &c. are all equal; and ∴ the ∠s subtended by them at the centres are all equal to one another.

Hence, if each of the ∠s AOM_1, M_1OM_2, &c., CPN_1, N_1PN_2, &c. be denoted by a, we have

$$\left.\begin{aligned} &\text{the } \angle AOB = ma, \\ \text{and } &\text{the } \angle CPD = na; \end{aligned}\right\}$$

$$\therefore \quad \frac{\text{the } \angle AOB}{\text{the } \angle CPD} = \frac{m}{n},$$

$$\text{and is } \therefore \quad = \frac{\text{the arc } AB}{\text{the arc } CD}. \qquad \text{Q. E. D.}$$

(ii) Let the ∠s AEB, CFD at the circumferences stand on the arcs AB, CD.

To prove that $\dfrac{\text{the } \angle AEB}{\text{the } \angle CFD} = \dfrac{\text{the arc } AB}{\text{the arc } CD}$.

Proof. Join AO, OB, CP, PD.

$$\left.\begin{aligned} \text{Then the } \angle AOB &= \text{twice the } \angle AEB \\ \text{and the } \angle CPD &= \text{twice the } \angle CFD. \end{aligned}\right\}$$

$$\therefore \quad \frac{\text{the } \angle AOB}{\text{the } \angle AEB} = \frac{\text{the } \angle CPD}{\text{the } \angle CFD} \text{ (each being} = 2)$$

and ∴ alternately,

$$\frac{\text{the } \angle AOB}{\text{the } \angle CPD} = \frac{\text{the } \angle AEB}{\text{the } \angle CFD}.$$

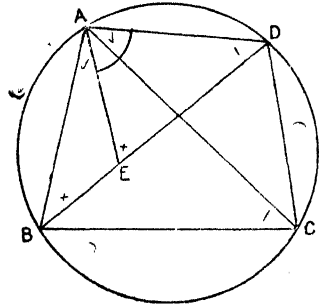
$$\text{But} \quad \frac{\text{the } \angle AOB}{\text{the } \angle CPD} = \frac{\text{the arc } AB}{\text{the arc } CD} \text{ (\textit{Proved above})};$$

$$\therefore \quad \frac{\text{the } \angle AEB}{\text{the } \angle CFD} = \frac{\text{the arc } AB}{\text{the arc } CD}. \qquad \text{Q. E. D.}$$

Note. From Book III, (Th. 4, Cor. 3, it is known that in equal circles sectors standing on equal arcs are equal. Hence it is easy to see that the sector **AOB** : the sector **CPD** = the arc **AB** : the arc **CD**.

Theorem 12. (Euc. VI. D.)

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quadl. inscribed in a ⊙, and let AC, BD be its diagonals.

To prove that AC.BD = AB.DC + AD.BC.

Proof. Make the ∠ BAE = the ∠ CAD, and let AE meet BD in E.

Then in the △s BAE, CAD,

the ∠ BAE = the ∠ CAD ;

and the ∠ ABE = the ∠ ACD. (*∠s in the same segment.*)

Hence the two △s are equiangular;

$$\therefore \frac{AB}{AC} = \frac{BE}{DC}; \qquad (Th.\ 3.)$$

and ∴ AB.CD = AC.BE. (*Sec. I, Art, 18.*) ... (*a*)

Again, adding the ∠EAC to each of the equal ∠s BAE, CAD, we have

the ∠BAC = the ∠EAD;

also the $\angle$ ACB = the $\angle$ ADE.

($\angle$*s in the same segment.*)

Hence the $\triangle$s ACB, ADE are equiangular;

$$\therefore \quad \frac{AD}{AC} = \frac{DE}{BC},$$

and $\therefore$ AD.BC = AC.DE. ... ... (β)

Hence, from (α) and (β),

AB.DC + AD.BC = AC.BE + AC.DE

= AC(BE + DE) (*Bk. II, Th.* 7).

= AC.BD. Q. E. D.

Note. This proposition is known as **Ptolemy's Theorem.**

EXERCISE (33).

1. ABC is an equilateral triangle inscribed in a given circle, and P is any point on the arc BC. Prove that PA = PB + PC.

2. If the diagonals of a cyclic quadrilateral be perpendicular to each other, prove that the sum of the two rectangles contained by its opposite sides is double the area of the quadrilateral.

3. ABC is a triangle; AD is the perpendicular from A on BC, and AP is a diameter of the circum-$\odot$ of the given triangle Prove that AP.AD = AB.AC.

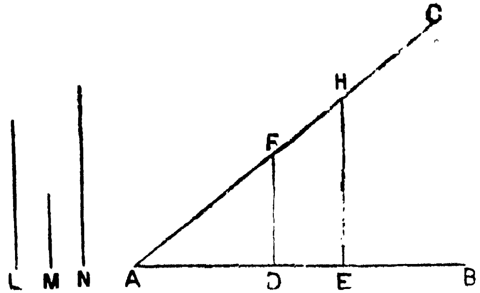
Hence, if a, b, c, γ be the numerical measures of the sides and the circum-radius respectively of any given triangle, and if S be the numerical measure of its area, prove that $\gamma = abc \div 4S$.

SECTION III.

Problems.

Problem 1.

To find the fourth proportional to three given straight lines.



Let L, M, N be the three given str. lines.

It is required to find a str. line X such that L : M = N : X.

Cons. Draw two str. lines AB, AC, forming an angle and unlimited towards B and C.

From AB cut off AD = L, and DE = M ; and from AC cut off AF = N.

Join DF ; and draw EH ∥ to DF, meeting AC at H.

Then FH is the reqd. str. line.

Proof. In the $\triangle$AEH, DF is $\parallel$ to EH ;

$$\therefore \qquad \frac{AD}{DE} = \frac{AF}{FH}, \qquad (Th\ 2.)$$

But AD = L, DE = M, and AF = N.

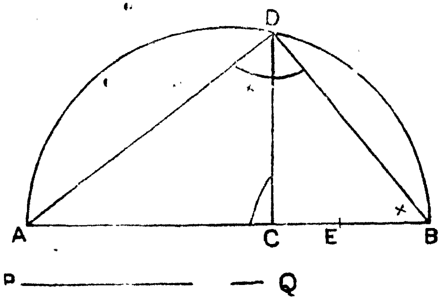
$$\therefore \qquad \frac{L}{M} = \frac{N}{FH};$$

i.e., FH is the fourth proportional to L, M, N. Q. E. F.

Note. By a similar construction we can find the *third* proportional to two given straight lines. For, if L, M be the two given straight lines, the only change in the construction will be to make **AF = M** ; and then **FH** will be such that **L : M = M : FH.**

Problem 2.

To find the mean proportional to two given straight lines.



Let AB, PQ be the two given straight lines.

It is required to find a str. line X such that AB : X = X : PQ.

Cons If AB is the greater of the two given str. lines, from AB cut off AC = PQ ; and on AB as diameter describe a semi-⊙.

Draw CD ⊥ to AB meeting the semi-⊙ at D.

Join AD, and from AB cut off AE = AD.

Then AE is the reqd. str. line.

Proof. Join DB.

The ∠ ADB, being in a semi-⊙, is a rt. ∠;

∴ the ∠ ADB = the ∠ ACD ;

also the ∠ ABD = the ∠ ADC, each of them being complementary to the ∠ BAD.

Hence the △s ADB, ACD are equiangular ;

∴ $\frac{AB}{AD} = \frac{AB}{AC}$. (*Th. 3.*)

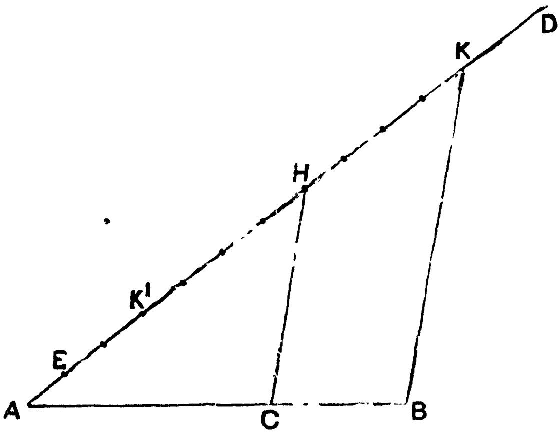
But AD = AE, and AC = PQ,

∴ $\frac{AB}{AE} = \frac{AE}{PQ}$;

i.e., AE is the mean proportional to AB and PQ. Q. E. F.

Problem 3.

To divide a given straight line internally in a given ratio (say in the ratio of 7 to 4).



Let AB be the given str. line.

It is required to find a pt. C in AB such that $\frac{AC}{CB} = \frac{7}{4}$.

Cons. Draw a str. line AD, making an angle with AB, and unlimited towards D.

From AD cut off any convenient length AE, and measure off AH = 7 times AE.

Along HD measure off HK = 4 times AE.

Join KB; and draw HC ∥ to KB, meeting AB in C.

Then C is the reqd. pt.

Proof. Evidently $\frac{AH}{HK} = \frac{7}{4}$.

Now in the $\triangle$ ABK, CH is $\parallel$ to BK;

$$\therefore \frac{AC}{CB} = \frac{AH}{HK} \qquad (Th.\ 2.)$$

$$= \frac{7}{4}. \qquad \text{Q. E. F.}$$

Note 1. If **HK′** equal to 4 times **AE**, be measured off along **HA**; if **K′B** be joined; and if **HC′** be drawn parallel to **K′B**, meeting **AB** produced in **C**; then **AB** is divided ***externally*** at **C** in the ratio of 7 to 4. This can be easily proved as above.

Note 2. **AH** = 7 times **AE**, and **AK** = 11 times **AE**;

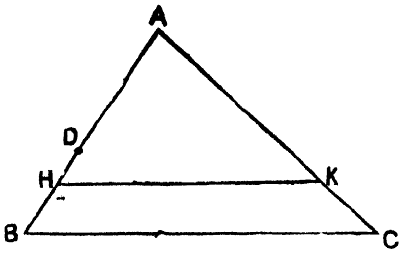
$\therefore \frac{AH}{AK} = \frac{7}{11}$. But the $\triangle$s **ACH, ABK** being equiangular, we have $\frac{AC}{AB} = \frac{AH}{AK}$. Hence $\frac{AC}{AB} = \frac{7}{11}$, *i.e.*, **AC** is $\frac{7}{11}$ of **AB**.

In the same way, it is easy to see that $AC' = \frac{7}{3}$ **AB**, where **C′** is found in the manner indicated in Note 1. Thus, by a construction similar to that of this Proposition, we can find *any* given fraction of a given straight line.

Note 3. There is only one point in which a given straight line may be divided internally (or externally) in a given ratio. For, if **C** be a point on **AB** such that $\frac{AC}{CB} = \frac{7}{4}$, then **AC** is $\frac{7}{11}$ of **AB**. Similarly, if **P** be a point on **AB** such that $\frac{AP}{PB} = \frac{7}{4}$, then **AP** is also $= \frac{7}{11}$ of **AB**. Hence **AP=AC**, and $\therefore$ **P** coincides with **C**.

Problem 4.

From a given triangle to cut off any given fraction (say $\frac{3}{5}$) of it by a straight line drawn parallel to one of its sides.



Let ABC be a given △, and let BC be the side considered.

It is required to draw a straight line HK parallel to BC so that the triangle AHK may be $=\frac{3}{5}$ of the triangle ABC.

Cons. Find a pt. D in AB such that

$$\frac{AD}{AB}=\frac{3}{5}. \qquad (Prob.\ 3.)$$

To AD and AB find the mean proportional AH, so that $AH^2 = AD.AB$. (*Prob. 2.*)

Draw HK ∥ to BC, meeting AC in K.

Then HK is the reqd. str. line.

Proof. The △s AHK, ABC are equiangular, and ∴ similar.

Hence, $\dfrac{\triangle AHK}{\triangle ABC} = \dfrac{AH^2}{AB^2}$ (*Th.* 7.)

$$= \frac{AD.AB}{AB^2}.$$

But, since 3 and 5 are the numerical measures of AC and AB, referred to one and the same unit, we have

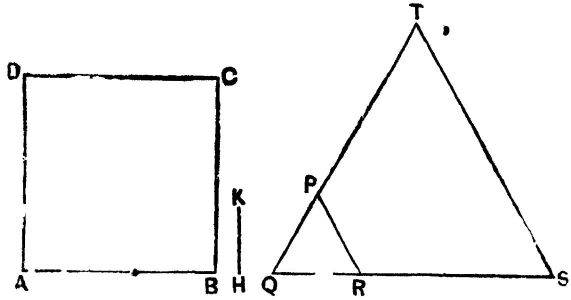
the rect. $AD.AB = 3 \times 5$ units of area,
and the sq. on $AB = 5 \times 5$ units of area;

$$\therefore \quad \frac{AD.AB}{AB^2} = \frac{3 \times 5}{5 \times 5} = \frac{3}{5}.$$

Hence, $\dfrac{\triangle AHK}{\triangle ABC} = \dfrac{3}{5}$. Q. E. F

Problem 5.

To construct an equilateral triangle equal to a given square.



Let ABCD be the given square.

It is required to construct an equilateral triangle equal to the sq. ABCD.

Cons. Describe an equilat. $\triangle PQR$. Find the str. line HK which is a side of the sq. equal to the $\triangle PQR$. (*Bk. II. Prob. 3, Note.*)

From QR, produced, if necessary, cut off QS = the fourth proportional to HK, AB, QR. (*Prob. 1.*)

Produce QP to T making QT = QS ; join TS.

Then TQS is the equilat. $\triangle$ reqd.

Proof. We have $\dfrac{HK}{AB} = \dfrac{QR}{QS}$; (*Cons.*)

$$\therefore \quad \frac{HK^2}{AB^2} = \frac{QR^2}{QS^2}. \qquad (Sec.\ 1,\ Art.\ 24.)$$

Now, the $\triangle$s PQR and TQS are similar ;

$$\therefore \quad \frac{\triangle PQR}{\triangle TQS} = \frac{QR^2}{QS^2} = \frac{HK^2}{AB^2}.$$

But $HK^2 = \text{the } \triangle PQR$; (*Cons.*)

$$\therefore \quad \frac{\triangle PQR}{\triangle TQS} = \frac{\triangle PQR}{\text{sq. } ABCD}.$$

Hence the equilat. $\triangle TQS = $ the sq. $ABCD$. Q. E. F.

EXERCISE (34).

1. On a given straight line construct a rectangle equal to a given rectangle.

2. Construct a square equal to a given rectangle.

3. Divide a given straight line internally in the ratio of 8 to 5.

4. Divide a given straight line externally in the ratio of 5 to 8.

5. AB is a given straight line. Find a straight line PQ such that

$$\frac{PQ^2}{AB^2} = \frac{2}{3}.$$

6. AB is a given straight line. Fine a straight line MN such that

$$\frac{MN^2}{AB^2} = \frac{9}{5}.$$

7. Trisect a given triangle by lines drawn parallel to its base.

8. Construct a triangle similar to a given triangle and equal in area to twice the given triangle.

9. $ABCD$ is a given quadrilateral, and PQ, a given straight line. On PQ construct a similar quadrilateral so that AB and PQ may be a pair of corresponding sides.

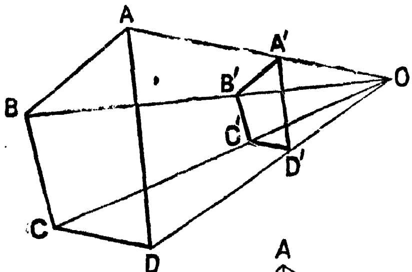
Hence shew how to construct a quadrilateral similar to the quadrilateral $ABCD$ and equal in area to one-third of it.

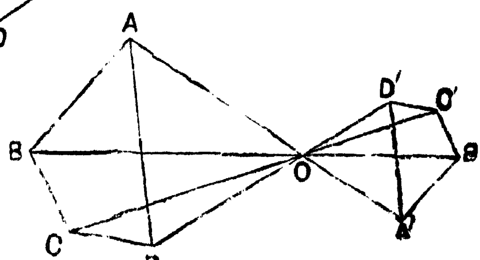
10. Construct a triangle, having given the base, the vertical angle and the ratio of the sides.

SECTION IV.

MISCELLANEOUS PROPOSITIONS.

1. *If two unequal similar figures are so placed that their corresponding sides are parallel, then the lines joining their corresponding angular points are concurrent.*





Let ABCD, A′B′C′D′ be two unequal similar figures, of which the angular points A′, B′, C′, D′ respectively correspond to the pts. A, B, C, D.

Let the figures be so placed that their corresponding sides are parallel.

To prove that AA′, BB′, CC′, DD′ are **concurrent.**

Proof. Let AA′, BB′ meet in O.

Then $A'B'$ being ∥ to AB, the △s AOB, $A'OB'$ are equiangular;

$$\therefore \quad \frac{BO}{B'O} = \frac{AB}{A'B'}.$$

Similarly, if CC' meets BB' in P, we have

$$\frac{BP}{B'P} = \frac{BC}{B'C'}.$$

But $$\frac{AB}{A'B'} = \frac{BC}{B'C'} \qquad (Hyp.)$$

$$\therefore \quad \frac{BO}{B'O} = \frac{BP}{B'P}.$$

and ∴ P coincides with O. (*Prob. 3, Note 3.*)

Thus, CC' passes through the pt. where AA' and BB' intersect, *i.e.*, through O.

In the same way it can be proved that DD' passes through the pt. where BB' and CC' intersect, and ∴ through O.

Hence, AA', BB', CC', DD' all pass through the same point O. Q. E. D.

Note. The point O is called a centre of similarity of the two figures $ABCD$, $A'B'C'D'$.

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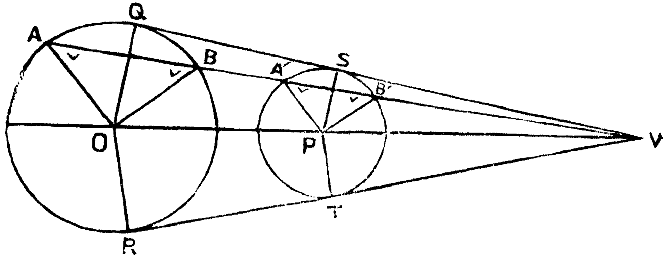
2. *If a direct common tangent to two given unequal circles meets the line of centres in* V, *then the other direct common tangent, as also the line joining the*

extremities of any two parallel radii of the circles drawn in the same sense, will pass through V.



Let O, P be the centres of the given ⊙s and QS a direct common tangent which meets OP in V.

Let RT be the other direct common tangent ; and let OA, PA′ be *any* two parallel radii drawn in the same sense.

Join AA′.

To prove that RT and AA′ both pass through V.

Proof. The ∠s OQV, PSV being rt. ∠s, OQ is ∥ to PS ;

∴ the △s OQV, PSV are equiangular ;

$$\therefore \frac{OV}{PV} = \frac{OQ}{PS}.$$

Thus, V divides OP *externally* in the ratio of the radii of the ⊙s.

Similarly, if RT meets OP in H, it can be proved that H divides OP externally in the ratio of the radii.

Hence H is no other pt. than V ; (*Prob. 3, Note 3.*)

i.e., AT passes through V.

Again if AA' meets OP in K, the Δs $OA'K$, PAK are evidently equiangular ;

$$\therefore \quad \frac{OK}{PK} = \frac{OA'}{PA}.$$

Thus, K divides OP externally in the ratio of the radii ; and $\therefore$ K coincides with V.

Hence AA' passes through V. Q. E. D.

Note 1. If AA' cuts the circles again in B and B', it is easy to see that OB and PB' are parallel. Hence the Δs OBV, $PB'V$ are equiangular ; the Δs OAV, $PA'V$ are equiangular too.

Hence, $\dfrac{VA}{VA'} = \dfrac{VO}{VP} = \dfrac{VB}{VB'}$; $\therefore$ $VA.VB' = VA'.VB$.

Note 2. $VA'.VB' = VS^2$; (*Bk. III, Th. 17, Cor. 1.*)

$\therefore$ $\dfrac{VA'}{VS} = \dfrac{VS}{VB'}$. Again, the Δs VSP, VQO are equiangular and so are the Δs $VB'P$, VBO ;

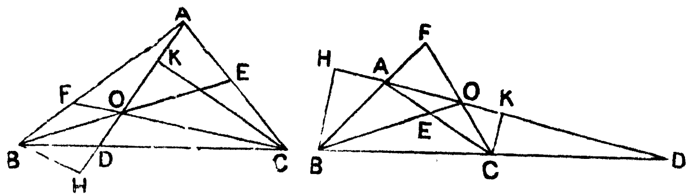
$\therefore$ $\dfrac{VS}{VQ} = \dfrac{VP}{VO} = \dfrac{VB'}{VB}$; $\therefore$ alternately, $\dfrac{VS}{VB'} = \dfrac{VQ}{VB}$.

Hence, $\dfrac{VA'}{VS} = \dfrac{VQ}{VB}$ $\therefore$ $VS.VQ = VA'.VB$.

Note 3. It can be easily proved, as in this Proposition, that *if a transverse common tangent to two given circles meets the line of centres in* W, *then the other transverse common tangent, as also the line joining the extremities of any two parallel radii of the circles drawn in opposite senses, will pass through* W.

Note 4. The point where the transverse common tangents to two given circles meet the line of centres is called the **internal centre of similitude** and the point where the direct common tangents meet the line of centres is called the **exrernal centre of similitude** of the two circles. These two points respectively divide the line joining the centres *internally* and *externally* in the ratio of the radii of the circles.

3. *If three concurrent straight lines are drawn from the vertices of a triangle to meet the opposite sides, then the product of three alternate segments, taken in order, is equal to the product of the other three.*



Let ABC be a $\triangle$, and let AD, BE, CF which intersect at O, meet the opp. sides at D, E, F.

Let the numerical measures of the segments (BD, DC), (CE, EA), (AF, FB) referred to one and the same unit be (p, p'), (q, q'), (r, r') respectively.

To prove that $p \cdot q \cdot r = p' \cdot q' \cdot r'$.

Proof. Draw BH, CK $\perp$ to AD.

Let α, β, γ be the numerical measures of the areas of the $\triangle$s BOC, COA, AOB respectively.

Now, the $\triangle$s BHD, CKD are equiangular ;

$$\therefore \qquad \frac{BD}{DC} = \frac{BH}{CK}.$$

Also the $\triangle$s AOB, AOC have the same base AO, and their altitudes are BH, CK respectively ;

$$\therefore \qquad \frac{\triangle AOB}{\triangle AOC} = \frac{BH}{CK}.$$

Hence, $\dfrac{BD}{DC} = \dfrac{\triangle AOB}{\triangle AOC}$,

i.e., $\dfrac{p}{p'} = \dfrac{\gamma}{\beta}$. ... ... (1)

Similarly, $\dfrac{q}{q'} = \dfrac{\alpha}{\gamma}$; ... ... (2)

and $\dfrac{r}{r'} = \dfrac{\beta}{\alpha}$. ... ... (3)

Hence, from (1), (2) and (3),

$$\frac{p}{p'} \cdot \frac{q}{q'} \cdot \frac{r}{r'} = \frac{\gamma}{\beta} \cdot \frac{\alpha}{\gamma} \cdot \frac{\beta}{\alpha}$$

$$= 1;$$

$\therefore$ $p \cdot q \cdot r = p' \cdot q' \cdot r'$. Q. E. D.

Note 1. This Proposition is known as **Ceva's Theorem.**

Note 2. The converse of this Proposition is the following: *If three straight lines* **AD**, **BE**, **CF** *drawn from the vertices of a* $\triangle$**ABC**, *meet the opposite sides in* **D**, **E**, **F**, *so that* **BD.CE.AF = DC.EA.FB**; *then the three straight lines* **AD**, **BE**, **CF** *are concurrent.* It can be proved as follows:—

Let **AD**, **BE** intersect at **O**; and let **CO** produced meet **AB** in **F′**. Let (p, p'), (q, q'), (r, r'), (s, s') be the numerical measures of (**BD**, **DC**), (**CE**, **EA**), (**AF**, **FB**) and (**AF′**, **F′B**) respectively, referred to one and the same unit.

Then, as before,

$$p \cdot q \cdot s = p' \cdot q' \cdot s'.$$

But $p \cdot q \cdot r = p' \cdot q' \cdot r'$. (*Hyp.*)

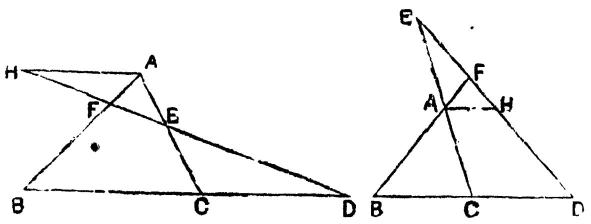
$\therefore$ $\dfrac{s}{r} = \dfrac{s'}{r'}$, and $\therefore$ $\dfrac{s}{s'} = \dfrac{r}{r'}$.

Hence, $\dfrac{AF'}{F'B} = \dfrac{AF}{FB}$;

i.e., **F′** and **F** both divide **AB** internally (or externally) in the same ratio; $\therefore$ **F′** coincides with **F** (Prob. 3, Note 3). Thus, **CO** produced

passes through F, which proves that **AD**, **BE**, **CF** all pass through the same pt. **O**.

4. *If a straight line cuts the sides, or the sides produced, of a triangle, the product of three alternate segments taken in order is equal to the product of the other three.*



Let ABC be a $\triangle$; and let a str. line cut the sides BC, CA, AB, or these sides produced, at D, E, F respectively.

Let the numerical measures of (BD, DC), (CE, EA), (AF, FB) referred to one and the same unit be (p, p'), (q, q'), (r, r') respectively.

To prove that $p.q.r = p'.q'.r'$.

Proof. Draw AH $\|$ to BC, meeting the str. line DEF in H ; and let m be the numerical measure of AH referred to the unit already used.

The two $\triangle$s BDF, AHF are equiangular ;

$$\therefore \quad \frac{BD}{AH} = \frac{BF}{AF};$$

$$\therefore \quad \frac{p}{m} = \frac{r'}{r}. \quad \dots \quad \dots \quad \dots \quad \dots \quad (1)$$

Also the two $\triangle$s AHE, CDE are equiangular ;

$$\therefore \frac{AH}{CD} = \frac{AE}{EC};$$

$$\therefore \frac{m}{p'} = \frac{q'}{q}. \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

Hence, from (1) and (2),

$$\frac{p}{q'} = \frac{r'}{r} \cdot \frac{q'}{q};$$

$$\therefore p.q.r = p'.q'.r'. \qquad \text{Q. E. D.}$$

Note 1. This proposition is known as **Menelaus Theorem.**

Note 2. The converse of this proposition is the following :—

If in a triangle, three points are taken, one on each of two sides and one on the produced part of the third side, (or one on the produced part of each of the three sides), so that the product of three alternate segments taken in order is equal to the product of the other three, then the three points are collinear. The first case can be proved as follows :—

In the $\triangle$**ABC**, let **D** be taken on **BC** produced and **E**, **F** on **CA** and **AB** respectively, so that **BD.CE.AF = DC.EA.FB** ; *to prove that the points* **D**, **E**, **F** *are collinear.*

Join **DE** and produce it to meet **AB** in **F′**. Let the numerical measures of (**BD**, **DC**), (**CE**, **EA**), (**AF**, **FB**), (**AF′**, **FB′**), referred to one and the same unit, be respectively (p, p'), (q, q'), (r, r'), (s, s').

Then, as before,

$$p.q.s = p'.q'.s'.$$

But $\quad p.q.r = p'.q'.r'. \qquad$ (*Hyp.*)

$$\therefore \frac{s}{r} = \frac{s'}{r'}, \quad \text{and} \quad \therefore \frac{s}{s'} = \frac{r}{r'}.$$

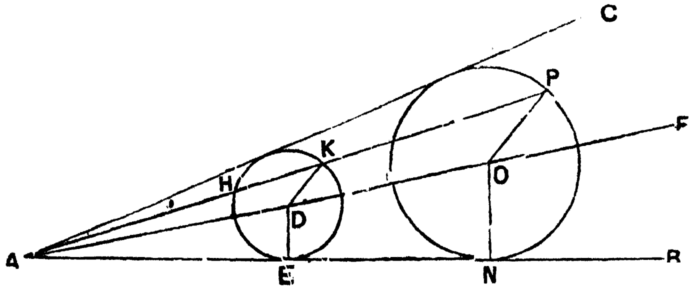
Hence, $\dfrac{AF'}{F'B} = \dfrac{AF}{FB}$

i.e. **F′** and **F** both divide **AB** *internally* in the same ratio ; $\therefore$ **F′** coincides with **F.**

Hence, DE produced passes through F. Q. E. D.

Similarly the other case can be proved.

5. *Construct a circle passing through a given point* P *and touching two given straight lines* AB, AC.



Cons. Let AF be the bisector of the ∠ BAC; then the centre of the reqd. ⊙, being equidistant from AB and AC, must be on AF.

Take eny pt. D on AF, and draw DE ⊥ to AB.

With centre D and radius DE describe a ⊙; it will touch AB and AC.

Join AP cutting the above ⊙ in H and K.

Join KD; and draw PO ∥ to KD, meeting AF in O.

Then O is the centre of the reqd. ⊙.

Proof. Draw ON ⊥ to AB.

Then the △s AED, ANO are equiangular;

$$\therefore \quad \frac{ON}{DE} = \frac{OA}{DA}.$$

Also the $\triangle$s ADK, AOP are equiangular ;

$$\therefore \quad \frac{OA}{DA} = \frac{OP}{DK}.$$

Hence, $$\frac{ON}{DE} = \frac{OP}{DK};$$

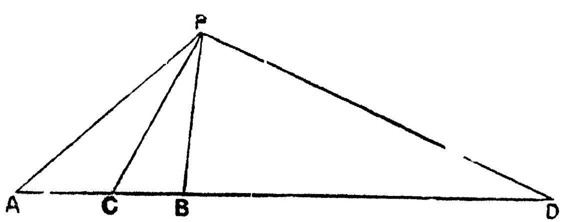
$$\therefore \quad \frac{ON}{OP} = \frac{DE}{DK}.$$

But DE = DK, $\therefore$ ON = OP.

Hence, the $\odot$ described with O as centre and ON as radius is the reqd. $\odot$. Q. E. F.

Note. If HD were joined and if PO′ were drawn parallel to HD, meeting AF in O′, then O′ would be the centre of another circle satisfying the given conditions.

6. *Given the base of a triangle and the ratio of the two sides ; shew that the different positions of the vertex all lie on the circumference of a known circle.*



Let AB be the given base, and let the ratio of the sides be $= \frac{m}{n}$, where m and n are integers.

Let P be *any* position of the vertex, so that $\frac{AP}{PB} = \frac{m}{n}$.

To prove that P lies on the circumference of a known circle.

Proof. Join PA, PB ; and let PC, PD be the internal and external bisectors of the $\angle$ APB, meeting AB in C and D.

Then the $\angle$ CPD is a rt. $\angle$.

Now, since PC is the internal bisector of the $\angle$ APB,

$$\therefore \frac{AC}{CB} = \frac{AP}{PB} = \frac{m}{n};$$

also, since PD is the external bisector of the $\angle$ APB,

$$\therefore \frac{AD}{DB} = \frac{AP}{PB} = \frac{m}{n}.$$

Thus, C and D divide AB internally and externally in the ratio of m to n, and are $\therefore$ known pts.

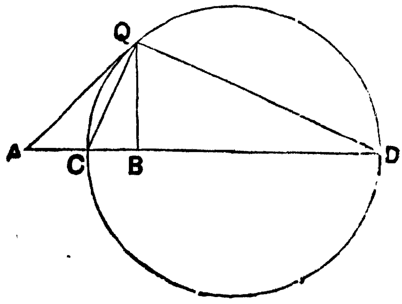
Now, since the $\angle$ CPD is a rt. $\angle$, the $\odot$ described on CD as diameter will pass through P.

Hence the $\odot$ described on CD as diameter is the $\odot$ on which P as well as every other position of the vertex lie.

Q. E. D.

Note. Take *any* point **Q** on the circle described on **CD** as diameter ; join **QA, QB, QC, QD.**

Now, $\frac{AC}{CB} = \frac{AD}{DB}$; each being $= \frac{m}{n}$; and **CD** subtends a right angle at **Q.** Hence, it can be proved, as in Th. 10, Note 1, that **QC** bisects the angle **AQB** ; and $\therefore$ $\frac{AQ}{QB} = \frac{AC}{CB} = \frac{m}{n}$.



Hence, Q is one of the positions of the vertex of the given triangle.

Now it is clear that *every* position of the vertex of the given triangle is on the circle described on **CD** as diameter, and conversely, *every* point on this circle is a position of the vertex. Hence, we may say that the *locus* of the vertex of the given triangle is the circle described on **CD** as diameter.

EXERCISE (35).

1. ABCD is a quadrilateral, and S is a point outside it. A′ is a point in AS such that AA′ : A′S = 2 : 1, B′, C′, D′ are points in SB, SC, SD such that A′B′, B′C′, C′D′ are parallel to AB, BC, CD respectively. Prove that the area of the figure A′B′C′D′ is one-ninth of the area of the given quadrilateral.

2. Construct a quadrilateral similar to a given quadrilateral so that each side of the constructed figure may be twice the corresponding side of the given figure.

3. ABC is a triangle, D and F being any two points in BC, AB respectively. DEF is a triangle, the point E being on the side of DF remote from B. If BE, or BE produced, intersects CA in E′, and if E′D′, E′F′ be drawn parallel to ED, EF, meeting BC, AB in D′, F′ respectively, prove that the triangle D′E′F′ is similar to the triangle DEF.

4. In a given triangle inscribe a triangle similar to another given triangle.

5. Prove that the internal and external *centres of similitude* of two given circles respectively divide the line joining the centres internally and externally in the ratio of the radii of the circles.

6. If a circle touches two given circles, prove that the line joining the points of contact passes through a centre of similitude of the two circles.

7. Prove that the orthocentre and the centroid of a triangle are respectively the external and internal centres of similitude of the circumcircle and nine-point circle of the triangle.

8. Prove that the straight lines which join the vertices of a traingle to the points of contact of the in-circle are concurrent.

9. The in-circle of a triangle ABC touches the sides BC, CA, AB at D, E, F. EF, FD, DE produced meet BC, CA, AB respectively in L, M, N. Prove that L, M, N are collinear.

10. From a given point O a straight line is drawn meet a given circle in P ; in OP a point Q is taken so that OP is to OQ in a given ratio, Find the locus of Q.

APPENDIX.

MODEL PAPERS.

MATRICULATION EXAMINATION.

(1) Compulsory Course.

[*Time allowed for answering each paper*—$1\frac{1}{2}$ *hours.*]

I.

1. If two straight lines intersect, prove that the vertically opposite angles are equal.

Shew that the bisectors of two vertically opposite angles are in one and the same straight line.

2. At a given point in a given straight line construct an angle equal to a given angle, using Ruler and Compasses.

3. Prove that parallelograms on the same base and of the same altitude are equal in area.

Deduce that the area of a parallelogram is equal to that of a rectangle contained by the base and altitude of the parallelogram.

4. Prove that there is one circle, and one only, which passes through three given points not in a straight line.

Deduce that one circle cannot cut another in more than two points,

5. If two chords of a circle intersect inside the circle prove that the rectangle contained by the parts of the one,

is equal to the rectangle contained by the parts of the other.

If two straight lines AB, CD intersect at P so that AP. PB = CP. PD, prove that the four points A, B, C, D are concyclic.

II.

1. Prove that two triangles are equal in every respect if the three sides of one triangle are respectively equal to the three sides of the other.

Prove that the straight line drawn from the vertex of an isosceles triangle to the mid-point of the base is perpendicular to the base.

2. Shew that the locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

Deduce that the point of intersection of the bisectors of any two angles of a triangle is equidistant from the three sides of the triangle.

3. Construct a rectangle equal to a given triangle using the necessary instruments. Also prove the construction.

4. Prove that triangles on the same base and of the same altitude are equal in area.

Hence shew that triangles on equal bases and of equal altitudes are equal in area.

5. If the line joining two points subtends equal angles at two other points on the the same side of it, prove that the four points lie on a circle.

Shew that triangles standing on the same base, and on the same side of it, with equal vertical angles have their vertices on the circumference of a circle of which the given base is a chord.

III.

1. Prove that the sum of the angles of a triangle is equal to two right angles.

If any side of a triangle is produced, shew that the exterior angle is equal to the sum of the two interior opposite angles.

2. If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, prove that the corresponding intercepts on any other straight line that cuts them are equal.

If D be the mid-point of the side AB of a triangle ABC, shew that the straight line drawn through D parallel to BC will bisect AC.

3. Shew that the locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

Deduce that the point of intersection of the perpendicular bisectors of any two sides of a triangle is equidistant from the vertices of the triangle.

4. Divide a given straight line into seven equal parts, using the necessary instruments. Give reasons for the construction.

5. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

If a circle passes through the angular points of a parallelogram, shew that the paralellogram must be a rectangle.

IV.

1. Prove that two right-angled triangles are equal in every respect if they have their hypotenuses equal, and one side of the one equal to one side of the other.

OB and OC are perpendiculars upon two straight lines AB and AC which meet at A : if $OB = OC$, prove that AO is the bisector of the angle BAC.

2. Illustrate and explain the Geometrical theorem coresponding to the identity,

$$(a+b)^2 = a^2+2ab+b^2.$$

If P is the mid-point of a straight line AB, shew that $AB^2 = 4AP^2$.

3. Construct a triangle equal in area to any given quadrilateral, using the necessary instruments.

4. Prove that a straight line drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.

Hence shew that a straight line cannot cut a circle in more than two points.

5. If a straight line touch a circle, and from the point of contact a chord be drawn, prove that the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

If ABC is a triangle, and if through B, a straight line BD be drawn on the side of BC remote from A such that

the angle CBD is equal to the angle BAC, shew that BD must be the tangent at B to the circum-circle of the triangle ABC.

V.

1. Prove that any two sides of a triangle are together greater than the third.

Hence shew that the difference of any two sides of a triangle is less than the third side.

2. Illustrate and explain the Geometrical theorem corresponding to the identity $(a-b)^2 = a^2 - 2ab + b^2$.

If C is the mid-point of a given straight line AB, and if D is any point in CB, shew that $AD^2 + DB^2 = 2AC^2 + 2CD^2$.

3. Construct a rectangle equal to any given quadrilateral, using the necessary instruments.

4. Prove that equal chords of circle are equidistant from the centre. Hence shew that in a circle the middle points of equal chords are concyclic.

5. Prove that the angle in a semi-circle is a right angle.

Hence shew how to construct a tangent to a circle from any point outside it.

VI.

1. If the sides of a convex polygon are produced in order, prove that the sum of the angles so formed is equal to four right angles.

Hence find the magnitude of an angle of a regular pentagon.

2. Define *parallel straight lines*. If a straight line cutting two other straight lines makes the alternate angles equal, prove that those two straight lines are parallel.

Hence shew that if two straight lines are prependicular to the same straight line, they are parallel to one another.

3. Construct a tangent to a given circle from a given point outside it, using the necessary instruments.

4. Prove that the tangent at any point of a circle is perpendicular to the radius through the point.

If from the point of contact of a tangent to a circle a straight line be drawn perpendicular to the tangent, prove that it must pass through the centre of the circle.

5. If the opposite angles of a quadrilateral are supplementary, prove that a circle can be described through its angular points.

The side AB of a quadrilateral ABCD is produced to E; if the angle CBE be equal to the angle ADC, shew that the four points A, B, C, D are concyclic.

VII.

1. If a straight line cuts two parallel straight lines, prove that the alternate angles are equal.

Hence shew that if a straight line is perpendicular to one of two parallel straight lines, it is also perpendicular to the other.

2. If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, prove that the angle contained by those two sides is a right angle.

Hence show that a triangle whose sides are respectively 28, 45 and 53 units of length, is a right-angled triangle.

3. **Take a triangle; bisect any two of its angles; from the point where the two bisectors meet, draw perpendiculars to the sides of the triangle. Verify by actual measurement that these three perpendiculars are equal in length.**

4. If two tangents are drawn to a circle from an external point, prove that they are equal and that they subtend equal angles at the centre of the circle.

Two tangents are drawn to a circle at the extremities of a diameter; shew that the intercept between them of any third tangent subtends a constant angle at the centre.

5. If two chords of a circle intersect outside the circle, prove that the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

If from any point P without a circle two straight lines be drawn, one of which cuts the circle in A and B, and the other touches it at T, prove that $PA.PB = PT^2$.

VIII.

1. If two sides of a triangle are unequal, prove that the greater side has the greater angle opposite to it.

Hence if one angle of a triangle be greater than another prove that the side opposite to the greater angle is greater than the side opposite to the less.

2. Prove that in a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.

Shew how to construct a square whose area is treble that of a given square.

3. Take a triangle; draw perpendiculars to any two of its sides at their middle points ; join the angular points to the point of intersection of these two perpendiculars. Verify, by actual measurement, that the three straight lines so drawn are equal in length.

4. Prove that in a circle the perpendicular drawn from the centre to a chord bisects the chord.

Hence shew that chords which are equidistant from the centre are equal.

5. If two circles touch, prove that the point of contact lies on the straight line through the centres.

Two circles whose centres are O and O′ touch each other externally at P; APB is a straight line through P meeting the circles again at A and B. Prove that OA is parallel to O′B.

IX.

1 Prove that two triangles are equal in every respect if two angles and a side of the one triangle are respectively equal two angles and the corresponding side of the other.

If the bisector of the vertical angle of a triangle is also perpendicular to the base, shew that the triangle is isosceles.

2. Prove that, in an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the other two sides together with twice the rectangle contained by one of these two sides and the projection on it of the other

If the numerical measures of the sides of a triangle are 7, 9 and 11, shew that the angle opposite to the greatest side is neither right nor obtuse.

3. Through a given point construct a straight line parallel to a given straight line, using Ruler and Compasses. Give reasons for the construction.

4. Prove that, in equal circles, if two arcs subtend equal angles at the centre, they are equal.

Hence shew that in equal circles, sectors having equal angles are equal.

5. Prove that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Shew that, in equal circles, arcs which subtend equal angles at the circumferences are equal.

X.

1. If two sides of a triangle are equal, prove that the angles opposite to these sides are equal.

Shew that three angles of an equilateral triangle are equal to one another.

2. Construct a parallelogram, with Ruler and Compasses; and verify, by actual measurement, that its diagonals bisect each other.

3. Prove that in any triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by one of these two sides and the projection on it of the other.

The numerical measures of the sides of a triangle are 8, 13 and 17; shew that the angle opposite to the greatest side is neither right nor acute.

4. In two equal circles, if there be two equal arcs one in each, prove that the chords of these two arcs are equal.

Hence shew that in the same circle chords of any two equal arcs are equal.

5. Prove that angles in the same segment of a circle are equal.

If two triangles standing on the same base, on the same side of it, have equal vertical angles, shew that both the triangles will have the same circum-circle.

(2) Optional Course

[Time allowed for answering each Paper—1¼ hours.]

I.

1. Prove that the sum of the squares on any two side of a triangle is equal to twice the square on half the third side together with twice the square on the median that bisects third side.

ABCD is a rectangle; and P is any point, either inside or outside it. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

2. Construct an equilateral triangle in a given circle.

3. AD, BE are perpendiculars from the vertices A and B of a triangle on the opposite sides; BF is perpendicular to ED or ED produced. Prove that the angle FBD is equal to the angle EBA.

4. If two triangles are equiangular, prove that their corresponding sides are proportional.

In a right-angled triangle, if a perpendicular be drawn from the right-angle to the hypotenuse, prove that the triangles on each side of it are similar to the whole triangle and to one another.

II.

1. Prove that the tangent at any oint of a circle is perpendicular to the radius through the point.

If two circles are concentric, prove that all chords of the outer circle which touch the inner are equal.

2. Construct a circle, with a given radius, touching two given straight lines.

3. If two triangles standing on equal bases have their vertical angles supplementary, prove that their circum-circles are equal.

4. When are two polygons said to be *similar*? Illustrate your meaning by a diagram.

If a polygon is divided into triangles by lines joining a given point to its vertices, prove that any similar polygon can be divided into corresponding similar triangles.

III.

1. If two tangents are drawn to a circle from an external point, prove that they are equal and that they subtend equal angles at the centre of the circle.

If a quadrilateral be circumscribed about a circle, prove that the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

2. Construct a circle passing through a given point and touching a given circle at a given point.

3. Prove that the distance of each vertex of a triangle from the ortho-centre is double the distance of the circum-centre from the opposite side.

4. Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

ABCD is a quadrilateral, and the bisectors of the angles A and C meet on the diagonal BD. Prove that the bisectors of the angles B and D meet on AC.

IV.

1. If a straight line touch a circle, and from the point of contact a chord be drawn, prove that the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

Two circles intersect at A and B, and through P, any point in the circumference of one of them, the chords PA and PB are drawn to cut the other circle at C and D. Prove that CD is parallel to the tangent at P.

2. Construct a circle which shall touch a given circle, have its centre in a given straight line and pass through a given point in the given straight line.

3. AB is any chord of a given circle, and AD is a tangent to the circle at A. DPQ is a straight line parallel to AB, meeting the circumference in P and Q. Prove that the triangle PAD is equiangular to the triangle QAB.

4. If a straight line is drawn parallel to one of a triangle, prove that it cuts the other two sides proportionally.

ABCD is a trapezium of which AB, DC are the parallel sides. If any straight line EF parallel to AB cuts AD, BC in E and F respectively, prove that

$$AE : ED = BF : FC.$$

V.

1. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

Through each of the points of intersection of two circles, right lines are drawn, cutting one circle in A and B, and the other in C and D. Prove that AB is parallel to CD.

2. Construct three circles touching each other and having their centres at three given points which are not collinear.

3. AB is an arc of a circle and P its middle point; PM is any chord meeting the arc of the opposite segment in M and AB in N. Prove that $PM.PN = PA^2$.

4. If two triangles have their sides proportional, prove that they are also equiangular, having those angles equal which are opposite to corresponding sides.

State the conditions of similarity of two rectilineal figures, and shew that in the case of two triangles one of those conditions is sufficient.

VI.

1. Prove that the angle in a semi-circle is a right angle.

AD is a diameter of a circle circumscribed about any given triangle ABC, and L is the point of intersection of perpendiculars drawn from B and C on the opposite sides of the triangle. Prove that LD and BC bisect each other.

2. Construct a circle having a given radius and touching a given circle and a given straight line.

3. AB is a diameter of a circle, and AF any chord. C is any point in AB, and through C a straight line is drawn at right angles to AB, meeting AF, produced if necessary, at G and the circumference at D, prove that

$$FA.AG = BA.\ AC = AD^2.$$

4. Prove that the ratio of the areas of two similar triangles is equal to the ratio of the squares on the corresponding sides.

Shew that any given triangle is divided into four equal parts by the straight lines joining the mid-points of its sides.

VII.

1. Prove that the sum of the squares on the four sides of a quadrilateral is equal to the sum of squares on its diagonals together with four times the square on the line joining the mid-points of the diagonals.

2. If two chords of a circle intersect outside the circle, prove that the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

AO, BO are radii of a circle at right angles to each other. ACD is a straight line meeting OB in C and the circle in D. Prove that

$$AC. AD = 2\, OB^2.$$

3. Construct a regular hexagon about a given circle.

4. Prove that the ratio of the areas of two similar quadrilaterals is equal to the ratio of the squares on the corresponding sides.

VIII.

1. ABC is a triangle, and the exterior angles at B, C are bisected by lines BD, CD respectively, meeting in D. Prove that the angle BDC and half the angle BAC make up a right angle.

2. If two circles touch, prove that the point of contact lies on the straight line through the centres.

If the centres of two circles which touch each other externally be fixed, shew that the common tangents of the circles will touch another circle of which the straight line joining the fixed centres is the diameter.

3. Construct an equilateral triangle about a given circle.

4. If two triangles have one angle of the one equal to one angle of other, and the sides about these equal angles proportional, prove that the triangles are similar.

If two straight lines AB, CD intersect at P, so that AP : PC = PD : PB, shew that the triangles APD, CPB are similar.

IX.

1. State and prove the Geometrical theorem corresponding to the identity $(a-b)^2 = a^2 - 2ab + b^2$.

If C is the mid-point of a given straight line AB, and if D is any point in CB produced, prove that $AD^2 + DB^2 = 2AC^2 + 2CD^2$.

2. AB is a diameter of a circle, and C, D are any two points on the semi-cirle ACDB. The chords AD, BC intersect at E. Prove that the circum-circle of the triangle CDE will cut the given circle at right angles.

3. Construct a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

4. If two triangles be equiangular, prove that their circum-radii are in the ratio of any pair of corresponding sides.

Hence shew that the nine-point radius of any triangle is half the circum-radius.

X.

1. Shew that the angular points of regular pentagon are concyclic.

2. Prove that angles in the same segment of a circle are equal.

AB is a common chord of two circles; through C, any point on one of the circles, straight line CAD, CBE are drawn, terminated by the circumference of the other circle. Shew that the arc DE is invariable.

3. Bisect a triangle by a line drawn parallel to one of its sides.

4. A and B are two fixed points, and P is a variable point such that PA : PB = 3 : 2. Prove that P lies on the circumference of a fixed circle.

ANSWERS AND HINTS.

Exercise (1). [Pages 19—21.]

1. *A solid,* 2. *A solid,* 3. *Length.*

4. *A surface*; because it has length and breadth but *no thickness.*

5. *Only when* the two lines *coincide.*

6. The points A and B are in *each* plane; hence the straight line AB also lies wholly in *each* plane, and is therefore common to the two planes.

7. Because then they entirely coincide.

8. Because when the paper is folded the two parts CA and CB coincide and are therefore *equal.*

9. When their vertices coincide and the arms of one fall upon those of the other.

10. When the paper is folded the angle COA coincides with the angle COB and is therefore equal to it; hence OC is perpendicular to AB.

11. Let AO be produced to any point B; when the revolving line falls upon OB it has turned through two right angles, and it turns through two right angles more when it rotates from the position OB into the position OA. Hence it altogether turns through *four* right angles.

12. With A as centre and with a radius equal to CD draw a circle, cutting AB at E. Then AE is the required part; for AE is a radius of the circle, and is therefore equal to CD.

13. Take any other point D and draw a straight line from A to D. With A as centre and with a radius equal to BC draw a circle cutting AD or AD produced (if necessary) at E. Then AE is the required straight line; for AE is the radius of the circle and is therefore equal to BC, and it is also drawn from A.

14. Because when the paper is folded the angle COA coincides with the angle COB and is therefore equal to it.

Exercise (4). [Pages 43—44.]

7. Produce BD to meet AC at E. 8. Produce BD to meet AC at E. Then, it can be shewn that AB + AC > BE + EC.

9. In proving the first part, apply the result of the last example.

Exercise (6). [Pages 57—58.]

2. Drawing a figure, it may be shewn that the greatest angle is equal to its supplement. 3. The greatest angle may be shewn to be greater than its supplement, and $\therefore$ it exceeds a rt. angle. 4. A diagonal divides it into two triangles. 7. 144.° 9. 36.° 13. The $\angle$ BCD may be shewn to be equal to its supplement.

Exercise (8.) [Pages 73—74.]

5. Join AA′, BB′, CC′. 9. Produce DE to F making EF = DE and join FC; then FC may be shewn to be equal and parallel to DB.

Exercise (11). [Pages 110—112.]

1. Produce AB to E, making AE = AC; join ED. Then CD = ED. 2. Produce AD to E, making DE = AD. Then AC = BE. 4. ED is ∥ to BF. Now apply Th. 24, Cor. 1. 12. Draw DP ∥ to CA, meeting AB in P; take PG = PA. 15. Draw BH ∥ to DE, meeting AP produced in H, draw CK ∥ to DE, meeting AP in K.

Exercise (12). [Page 117.]

7. 12.

Exercise (13). [Pages 124—125.]

3. Through E draw a str. line ∥ to AB. 10. 4·9 sq. inches. 11. 2·5 inches. 12. 5·5 inches.

Exercise (14). [Pages 129—130.]

5. 13 miles. 6. 35 feet. 7. 1 foot.

8. At one extremity of the given line make an ∠ equal to half a right ∠. 9. At one end of the given line draw a ⊥ to it equal to a side of the given square. 12. Draw BD ⊥ to BC and = BA; join DC. Now apply Th. 21, Bk. I.

Exercise (15). [Pages 138—140.]

1. The square on any given line = four times the square on half the line. 2. The sq. on any given line = nine times the sq. on one-third of the line. 6. If a given str. line AB be divided into any three parts at C and D, then AD.CB = AC.DB + CD.AB. 7. If E is the mid-pt. of BC, AE = EB or EC. 8. If E is the mid-pt. of BC, then from the given condition we have

$EA^2 - ED^2 = EC^2 - ED^2$. **10.** Apply Theorems 10 and 11. **11.** Apply Theorems 10 and 11.

Exercise (16). [Pages 143—145.]

8. The circumference of a circle whose centre is the mid-pt. of AB and radius = 4 inches. **11.** If ABCD is the par^m., draw CE ∥ to BD, meeting AD produced in E.

Exercise (17). [Pages 151—152.]

10. Draw BH ⊥ to AB and = a side of the given square. Join AH. Make ∠AHF = the ∠HAF. **11.** Construct a square = the given rectangle, and then proceed as in the last problem.

Exercise (18). [Pages 160—162.]

2. O is the centroid of the △ABC. **6.** If AD be ⊥ to BC, it may be shewn that AC = 2CD. **9.** If AB is the given str. line, make ∠BAC = 45° and ∠ABC = $\frac{1}{2}$ of 45°. Draw CD ⊥ to AC, meeting AB in D. **14.** 84 units of area; 8. **15.** 324 units of area. **18.** If C is the mid-pt. of AB and PD is ⊥ to AB, it is easy to see that $PA^2 - PB^2 = 2AB.CD$. **21.** The centroid of the △.

Exercise (20). [Page 183.]

4. Draw PM ⊥ to AB, and produce PM to Q making MQ = PM. Then AP = AQ and BP = BQ.

Exercise (21). [Page 193.]

2. Construct an equilat. △ABC each of whose sides 2 inches; produce BC to D, making CD = BC. Then will be equal to a side of the reqd. △.

6. If O be the centre, the $\angle$ AOC is greater than the $\angle$ AOD.

Exercise (23). [Pages 209—210.]

9. It may be shewn that the $\angle APB = 90^\circ + \frac{1}{2}$ the $\angle$ ACB.

Exercise (25). [Pages 225—226.]

11. The $\angle ACD =$ the $\angle$ CBD, each being complementary to the $\angle$ BCD.

Exercise (26). [Page 262.]

6. 120°.

Exercise (27). [Pages 293—299.]

17. On the given side construct a segment containing an $\angle$. equal to $\frac{1}{2}$ the given $\angle$.

18. C is the mid-pt. of the arc PQ.

24. If P be on the arc AC, from PB cut off $PD = PC$.

25. EF may be shewn to be $\parallel$ to the tangent at A.

26. See Prop. 14.

27. Apply Prop. 14.

29. If O be the centre of the given $\odot$, it may be shewn that OC touches the $\odot$ CDE.

32. See Prop. 1.

33. See Prop. 10.

35. See Prop. 8. It is easy to see that OE touches the $\odot$SEQ.

38. B′C′ is $\parallel$ to BC and also to QR.

39. Apply Prop. 10.

41. The distance of either common tangent from the mid-point of the line joining the centres is = half the sum of the radii of the ⊙s.

42. The fixed pt. is the pt. where the bisector of the ∠ BAC meets the circum-⊙ of the △ABC. If O be this pt., it may be shewn that the ∠ POQ = ∠ BOC.

45. If PD be ⊥ to BC, D is the second pt. where the circles on PB, PC as diameters intersect.

46. If A be the fixed pt. and AK ⊥ to the fixed ∥ str. line, then the ⊙ with A as centre and AK as radius will be the fixed ⊙ reqd.

47. See Prop. 12, Note 2.

48. See Prop. 1.

49. In the figure of Prop. 15 it is easy to see that the nine-pt. ⊙ touches the ⊙ whose centre is A′ and radius = A′G.

50. Taking 1 inches = h miles, radius of the earth = R miles and the reqd. distance = X miles, it is easy to shew that $X^2 = (2R + h)h = 2Rh$, neglecting h^2 which is *very small* compared with $2Rh$.

Exercise (28). [Pages 317—318.]

1. 15. **7.** The ratio of A to B is equal to the ratio of C to D.

Exercise (32). [Page 317.]

6. Apply Addendo.

Exercise (33). [Pages 321—322.]

3. The two Δs ABD, APC are equiangular.

Exercise (34). [Pages 329—330.]

5. See Prob. 4. 6. See Prob. 4. 9. Divide AB at E so that AE : AB = 1 : 3, and then find PQ so that AE : PQ = PQ : AB.

10. The ratio of the segments into which the base is divided by the bisector of the vertical $\angle$ is known.

Exercise (35). [Pages 370—371.]

2. See the last example. 4. See the last example. 7. See Bk. III, Sec. IV. Prop. 15. 9. Apply Prop. 4 to each of the three lines EFL, DFM, DEN.